CompSci 275, Constraint Networks

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Constraint Optimization and counting
Chapter 13

Outline

- **Introduction**
  - Optimization tasks for graphical models
  - Solving optimization problems with inference and search
- **Inference**
  - Bucket elimination, dynamic programming
  - Mini-bucket elimination
- **Search**
  - Branch and bound and best-first
  - Lower-bounding heuristics
  - AND/OR search spaces

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Constraint optimization problems for graphical models

A *finite* COP is a triple $R = \langle X, D, F \rangle$ where:

$X = \{X_1, ..., X_n\}$ - variables

$D = \{D_1, ..., D_n\}$ - domains

$F = \{f_1, ..., f_m\}$ - cost functions

f(A,B,D) has scope {A,B,D}

A	B	D	Cost
1	2	3	3
1	3	2	2
2	1	3	0
2	3	1	0
3	1	2	5
3	2	1	0

Global Cost Function

$$F(X) = \sum_{i=1}^{m} f_i(X)$$

Constraint Optimization Problems for Graphical Models

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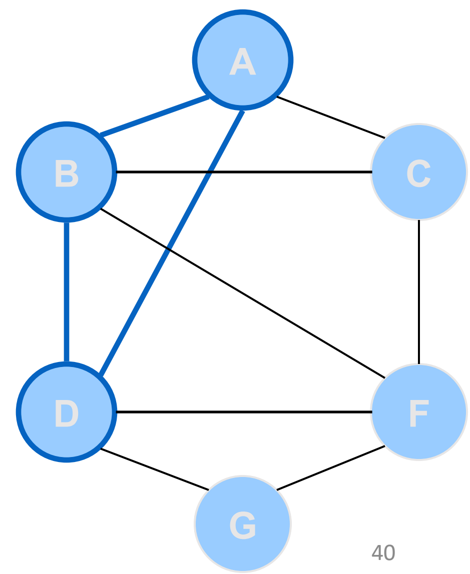
A	B	D	Cost
1	2	3	3
1	3	2	2
2	1	3	0
2	3	1	0
3	1	2	5
3	2	1	0

Primal graph =
Variables --> nodes
Functions, Constraints -→ arcs

F(A,B,C,D,F,G)= f1(A,B,D)+f2(D,F,G)+f3(B,C,F)

f1(A,B,D)
f2(D,F,G)
f3(B,C,F)

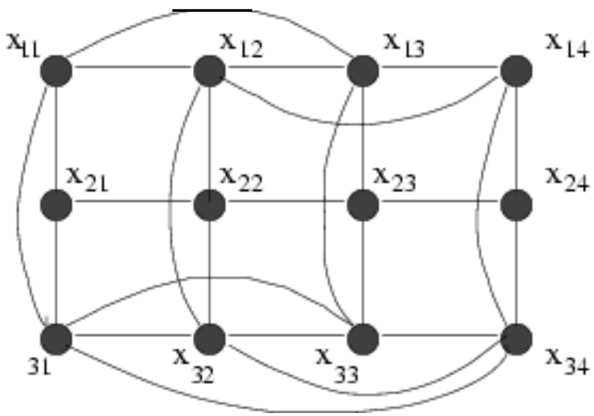


Global Cost Function

$$F(X) = \sum_{i=1}^{m} f_i(X)$$

Example: constrained optimization

Example: power plant scheduling



Unit #	Min Up Time	Min Down Time
1	3	2
2	2	1
3	4	1

Variables $= \{X_1,\ldots,X_n\}$, domain $= \{ON, OFF\}$.
Constraints : $X_1 \vee X_2, \neg X_3 \vee X_4$, min - up and min - down time,
power demand : $\sum \mathrm{Power}(X_i) \geq Demand$

Objective : minimize $\mathrm{TotalFuelCost}(\mathrm{X}_1,\ldots,\mathrm{X}_N)$

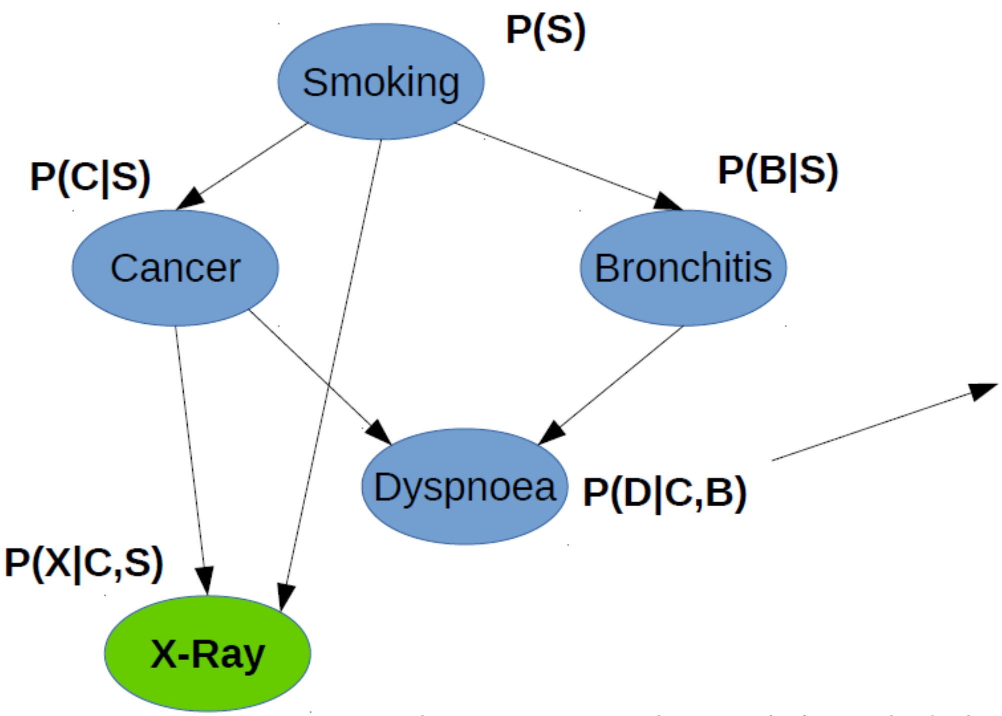
Example: combinatorial auction

Given a set of elements A= $\{a_1,\ldots, a_n\}$ and given a set of bids, B= $\{b_1,\ldots, b_l\}$, each is a subset of A, each having cost r_i, select a subset of bids B’ with maximal total cost where no two bids share an item.

$$max_{B\prime} Max \sum_{b_i \ in \ B\prime} r\_i$$

Consider a problem instance given by the following bids: $b_1 = \{1, 2, 3, 4, \}$, $b_2 = \{2, 3, 6\}$, $b_3 = \{1, 5, 4\}$, $b_4 = \{2, 8\}$, $b_5 = \{5, 6\}$ and the costs $r_1 = 8$, $r_2 = 6$, $r_3 = 5$, $r_4 = 2$, $r_5 = 2$. In this case the variables are b_1, b_2, b_3, b_4, b_5, their domains are $\{0, 1\}$ and the constraints are $R_{12}, R_{13}, R_{14}, R_{24}, R_{25}, R_{35}$. The cost network for this problem formulation is identical to its constraint network, since all the cost components are unary. The reader can verify that an optimal solution is given by $b_1 = 0, b_2 = 1, b_3 = 1, b_4 = 0, b_5 = 0$. Namely, selecting bids b_2 and b_3 is an optimal choice with total cost of 11. □

Probabilistic Networks



P(D|C,B)

C	B	D=1	D=0
0	0	0.1	0.9
0	1	0.7	0.3
1	0	0.8	0.2
1	1	0.9	0.1

$$P(S,C,B,X,D) = P(S)\cdot P(C|S)\cdot P(B|S)\cdot P(X|C,S)\cdot P(D|C,B)$$

MPE: Find a maximum probability assignment, given evidence

= Find $\arg\max P(S)\cdot P(C|S)\cdot P(B|S)\cdot P(X|C,S)\cdot P(D|C,B)$

Graphical models

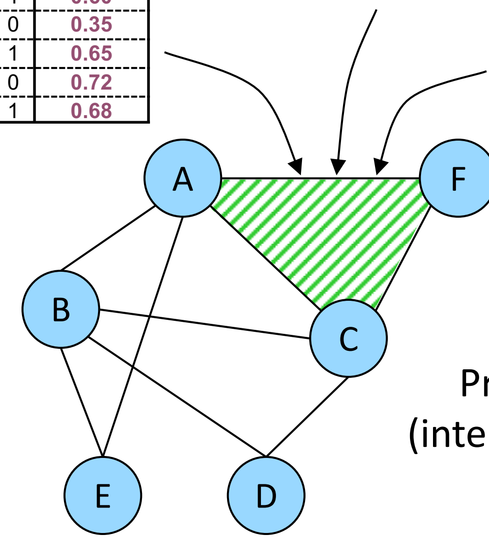
- A graphical model **(X,D,F)**:
  - **X** = $\{X_1,\ldots X_n\}$ variables
  - **D** = $\{D_1, \ldots D_n\}$ domains
  - **F** = $\{f_1,\ldots,f_m\}$ functions
- Operators:
  - combination
  - elimination (projection)
- Tasks:
  - **Belief updating**: $\Sigma_{X-y} \prod_j P_i$
  - **MPE**: $\max_X \prod_j P_j$
  - **CSP**: $\prod_X \times_j C_j$
  - **Max-CSP**: $\min_X \Sigma_j f_j$

A	C	F	P(F|A,C)
0	0	0	0.14
0	0	1	0.96
0	1	0	0.40
0	1	1	0.60
1	0	0	0.35
1	0	1	0.65
1	1	0	0.72
1	1	1	0.68

Relation

A	C	F
red	green	blue
blue	red	red
blue	blue	green
green	red	blue

$f_i \;:=\; (F = A + C)$



Primal graph
(interaction graph)

- All these tasks are NP-hard
  - exploit problem structure
  - identify special cases
  - approximate

Combination of cost functions

A	B	f(A,B)
b	b	6
b	g	0
g	b	0
g	g	6

+

B	C	f(B,C)
b	b	6
b	g	0
g	b	0
g	g	6

A	B	C	f(A,B,C)
b	b	b	12
b	b	g	6
b	g	b	0
b	g	g	6
g	b	b	6
g	b	g	0
g	g	b	6
g	g	g	12

= 0 + 6

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  - Mini-bucket elimination, belief propagation
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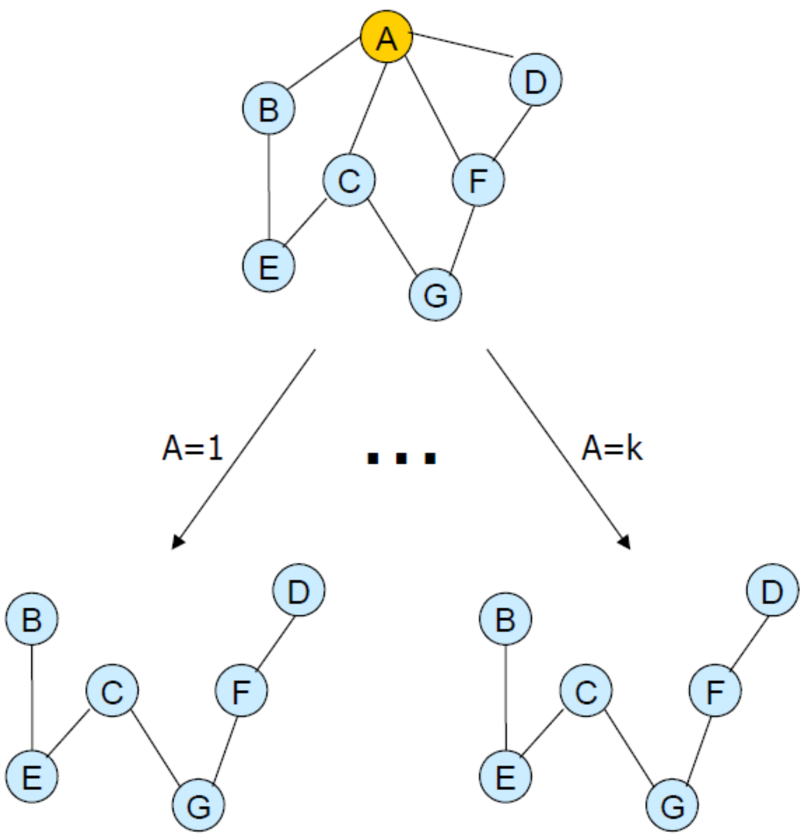
- **Hybrids of search and inference**
  - Cutset decomposition
  - Super-bucket scheme

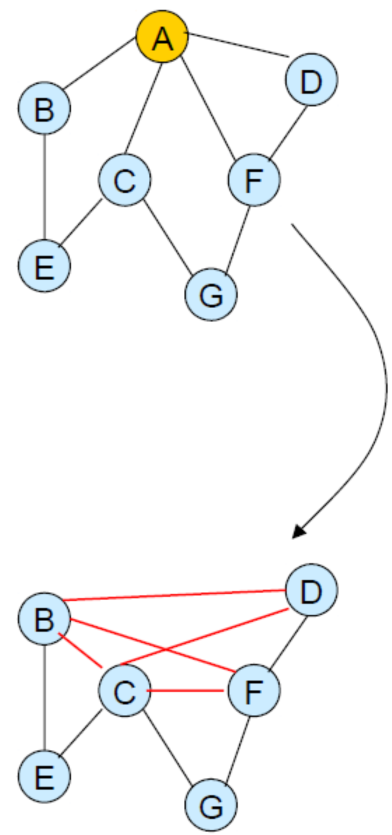
Conditioning vs. Elimination

Conditioning (search)



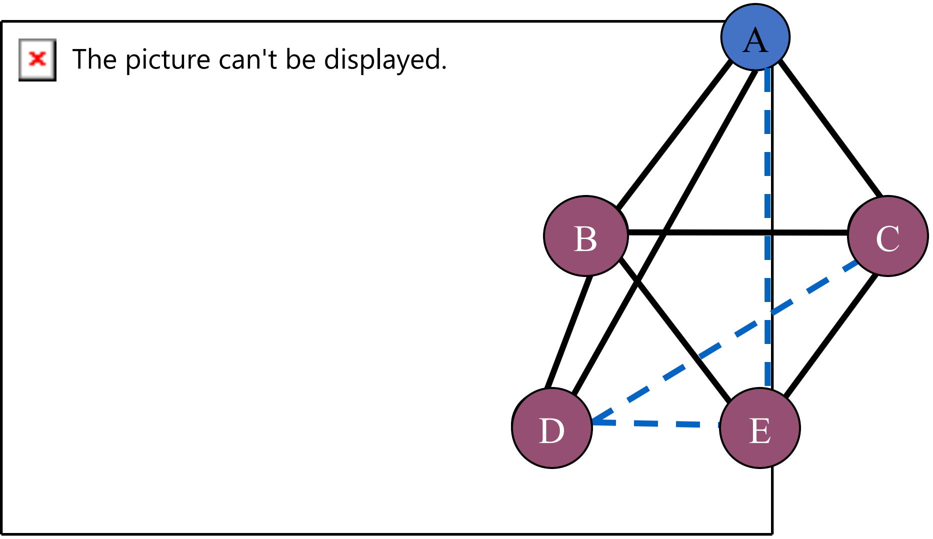
k “sparser” problems

Elimination (inference)



1 “denser” problem

Computing the optimal cost solution



Constraint graph

$$\textbf{OPT} = \min_{e=0,d,c,b} f(a,b)+f(a,c)+f(a,d)+f(b,c)+f(b,d)+f(b,e)+f(c,e)$$

Combination

$$\min_{e=0} \min_{d} f(a,d) + \min_{c} f(a,c)+f(c,e) + \min_{b} \underbrace{f(a,b)+f(b,c)+f(b,d)+f(b,e)}_{h^B(a,d,c,e)}$$

Variable Elimination

Finding $OPT = \min_{X_1,\ldots,X_n} \sum_{j=1}^{r} f_j(X)$

Algorithm **elim-opt** (Dechter, 1996)
Non-serial Dynamic Programming (Bertele and Briochi, 1973)

$$OPT = \min_{a,e,d,c,b} F(a,b) + F(a,c) + F(a,d) + F(b,c) + F(b,d) + F(b,e) + F(c,e)$$

$\min_{b} \sum$ ← Elimination operator

bucket B: f(a,b) f(b,c) f(b,d) f(b,e)

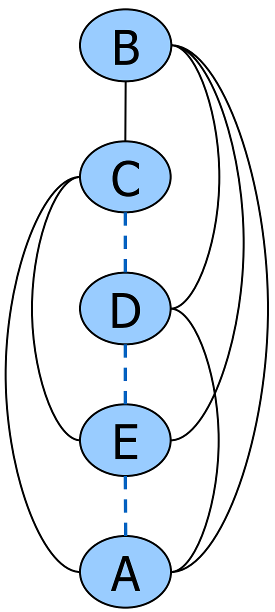
bucket C: f(c,a) f(c,e) ***$h^B(a,d,c,e)$***

bucket D: f(a,d) ***$h^C(a,d,e)$***

bucket E: e=0 ***$h^D(a,e)$***

bucket A: ***$h^E(a)$***

OPT



Generating the optimal assignment

5. $b' = \arg\min_b f(a',b) + f(b,c') + f(b,d') + f(b,e')$

4. $c' = \arg\min_c f(c,a') + f(c,e') + h^B(a',d',c,e')$

3. $d' = \arg\min_d f(a',d) + h^C(a',d,e')$

2. $e' = 0$

1. $a' = \arg\min_a h^E(a)$

B: f(a,b) f(b,c) f(b,d) f(b,e)

C: f(c,a) f(c,e) $h^B(a,d,c,e)$

D: f(a,d) $h^C(a,d,e)$

E: e=0 $h^D(a,e)$

A: $h^E(a)$

Return (a', b', c', d', e')

Complexity

Algorithm **elim-opt** (Dechter, 1996)
Non-serial Dynamic Programming (Bertele and Briochi, 1973)

$$OPT = \min_{a,e,d,c,b} F(a,b)+F(a,c)+F(a,d)+F(b,c)+F(b,d)+F(b,e)+F(c,e)$$

$\min_{b} \sum$ ← Elimination operator

bucket B: f(a,b) f(b,c) f(b,d) f(b,e)

bucket C: f(c,a) f(c,e) ***h^B(a, d, c, e)***

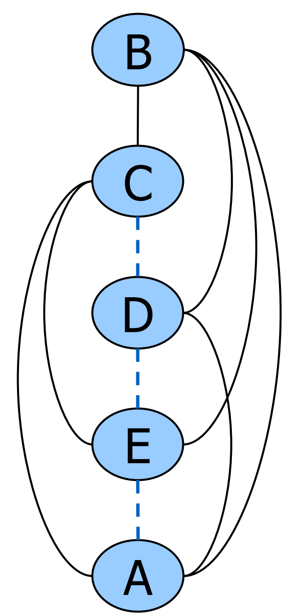
bucket D: f(a,d) ***h^C(a, d, e)***

bucket E: e=0 ***h^D(a, e)***

bucket A: ***h^E(a)***

OPT

exp(w*=4)
"induced width"
(max clique size)



Induced width

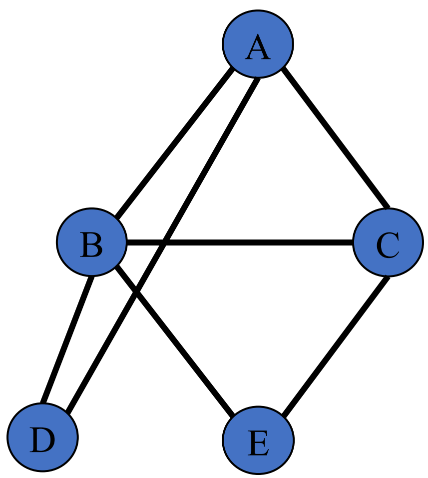
Bucket-elimination is **time** and **space**

$$O(r \exp(w^*(d))$$

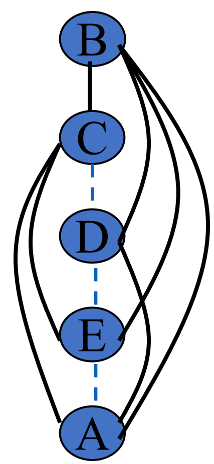
$w^*(d)$ – the induced width of the primal graph along ordering d

r = number of functions

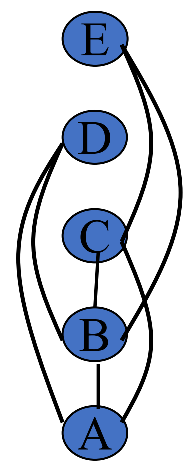
The effect of the ordering:



constraint graph



$w^*(d_1) = 4$



$w^*(d_2) = 2$

Finding smallest induced-width is hard

Using a different ordering

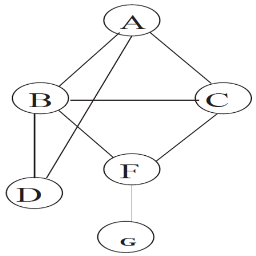


Figure 13.1: The cost graph of the cost function: $C(a,b,c,d,f,g) = F_0(a) + F_1(a,b) + F_2(a,c) + F_3(b,c,f) + F_4(a,b,d) + F_5(f,g)$

(a)

Bucket G: $F_5(g,f)$

Bucket D: $F_4(d,b,a)$

Bucket F: $F_3(f,b,c)$

Bucket B: $F_1(b,a)$

Bucket C: $F_2(c,a)$

Bucket A: $F_0(a)$

(b)

$\max \sum$

Bucket G: $F_5(f,g)$

Bucket D: $F_4(a,b,d)$

Bucket F: $F_3(b,c,f)$ $\quad h^G(f)$

Bucket B: $F_1(a,b)$ $\quad h^D(a,b)$ $\quad h^F(b,c)$

Bucket C: $F_2(a,c)$ $\quad h^B(a,c)$

Bucket A: $F_0(a)$ $\quad h^C(a)$

Figure 13.4: Bucket elimination along ordering $d_1 = A, C, B, F, D, G$.

Induced-width (again…)

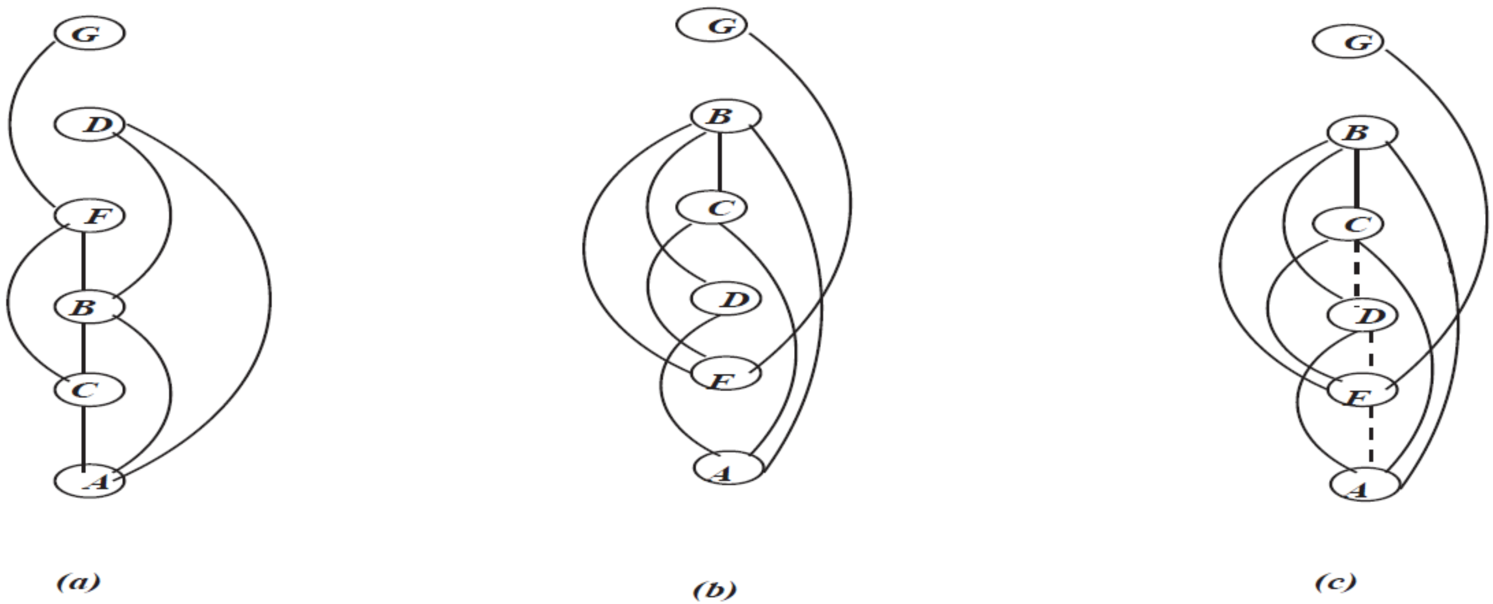


Figure 13.7: Two orderings of the cost graph of our example problem

Min sum
Is more popular

Elim-opt: BE for optimization

Algorithm elim-opt
Input: A cost network $\mathcal{C} = (X, D, C)$, $C = \{F_1, ..., F_l\}$; ordering d
Output: The maximal cost assignment to $\sum_j F_j$.
1. **Initialize:** Partition the cost components into ordered buckets.
2. **Process buckets** from $p \leftarrow n$ downto 1
For costs $h_1, h_2, ..., h_j$ defined over scopes $Q_1, ..., Q_j$ in $bucket_p$, do:

- **If** (observed variable) $x_p = a_p$, assign $x_p = a_p$ to each h_i and put in appropriate buckets. Terminate if value is inconsistent.
- **Else,** (sum and maximize)
  $A \leftarrow \cup_i Q_i - \{x_p\}$
  $h^p = max_{x_p} \sum_{i=1}^{j} h_i$.
  Place h^p in the latest lower bucket mentioning a variable in A.

3. **Forward:** From $i = 1$ to n, given $\vec{a}_{i-1}$, assign x_i a value a_i that maximizes the sum values of functions in its bucket.

Figure 13.5: Dynamic programming as elim-opt

Example: combinatorial auction

Given a set of elements A= $\{a_1,\ldots, a_n\}$ and given a set of bids, B= $\{b_1,\ldots, b_l\}$, each is a subset of A, each having cost r_i, select a subset of bids B’ with maximal total cost where no two bids share an item.

$$max_{B\prime} Max \sum_{b_i \text{ } in \text{ } B\prime} r\_i$$

Consider a problem instance given by the following bids: $b_1 = \{1,2,3,4,\}$, $b_2 = \{2,3,6\}$, $b_3 = \{1,5,4\}$, $b_4 = \{2,8\}$, $b_5 = \{5,6\}$ and the costs $r_1 = 8$, $r_2 = 6$, $r_3 = 5$, $r_4 = 2$, $r_5 = 2$. In this case the variables are b_1, b_2, b_3, b_4, b_5, their domains are $\{0,1\}$ and the constraints are $R_{12}, R_{13}, R_{14}, R_{24}, R_{25}, R_{35}$. The cost network for this problem formulation is identical to its constraint network, since all the cost components are unary. The reader can verify that an optimal solution is given by $b_1 = 0, b_2 = 1, b_3 = 1, b_4 = 0, b_5 = 0$. Namely, selecting bids b_2 and b_3 is an optimal choice with total cost of 11. □

Treating constraints as constraints

Algorithm elim-opt-cons
Input: A cost network $\mathcal{C} = (X, D, C_h, C_s)$, $C_h = \{R_{S_1}, ..., R_{S_m}\}$; $C_s = \{F_{Q_1}, ..., F_{Q_l}\}$. ordering d;
Output: A consistent solution that maximizes $\sum_{F_i \in C_s} F_i$.
1. **Initialize:** Partition the C_s and C_h into buckets using the usual rule.

2. **Process buckets** from $p \leftarrow n$ downto 1,
For costs $h_1, h_2, ..., h_j$ defined over scopes $Q_1, ..., Q_j$, for hard constraint relations $R_1, R_2, ..., R_t$ defined over scopes $S_1, ..., S_t$ in $bucket_p$, do:

- **If** (observed variable) $x_p = a_p$, assign $x_p = a_p$ to each h_i and each R_i and put in appropriate buckets.
- **Else,** (sum and maximize, join and project)
  1. Let $U_p = \cup_i S_i - \{x_p\}$, $V_p = \cup_i Q_i - \{x_p\}$, $W_p = U_p \cup V_p$
  2. $R^p = \pi_{U_p}(\bowtie_{i=1}^{t} R_i)$. (generate the hard constraint)
  3. For every tuple t over W_p do: (generate the cost function)
     $h^p(t) = max_{\{a_p|(t,a_p)\ satisfies\ \{R_1,...,R_l\}\}} \sum_{i=1}^{j} h_i(t, a_p)$.
     Place h^p in the latest lower bucket mentioning a variable in W_p. Place R^p in the bucket of the latest variable in U_p.

3. **Forward:** Assign maximizing values in ordering d, consulting functions in each bucket.

Figure 13.6: Algorithm elim-opt-cons

Elim-opt for auction problem

processing bucket b_4. In this bucket we compute $h^4(b_1, b_2) = \max_{\{(b_4|(b_1,b_2,b_4)\in R_{41}\bowtie R_{42}\}} r(b_4)$, yielding:

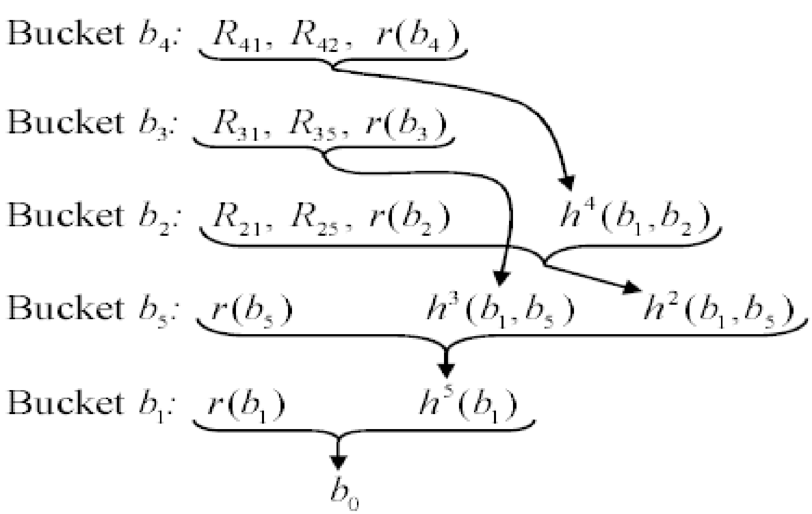
$$h^4(b_1,b_2) = \begin{cases} 0 & \text{if } b_1 = 1,\ or\ b_2 = 1 \\ 2 & \text{if } b_1 = 0, b_2 = 0 \end{cases}$$

Processing bucket b_3 we compute $h^3(b_1, b_5) = \max_{\{(b_3|(b_1,b_3,b_5)\in R_{31}\bowtie R_{35}\}} r(b_3)$, yielding:

$$h^3(b_1,b_5) = \begin{cases} 0 & \text{if } b_1 = 1,\ or\ b_5 = 1 \\ 5 & \text{if } b_1 = 0, b_5 = 0 \end{cases}$$

Processing bucket b_2, which now includes a new function, gives us:
$h^2(b_1, b_5) = \max_{\{(b_2|(b_1,b_2,b_5)\in R_{21}\bowtie R_{25}\}}(r(b_2) + h^4(b_1, b_2))$, yielding:

$$h^2(b_1,b_5) = \begin{cases} 0 & \text{if } b_1 = 1, b_5 = 1 \\ 0 & \text{if } b_1 = 1, b_5 = 0 \\ 2 & \text{if } b_1 = 0, b_5 = 1 \\ 6 & \text{if } b_1 = 0, b_5 = 0 \end{cases}$$



(a)

b1 = {1, 2, 3, 4}; *b2* ={2, 3, 6};
b3 = {1, 5, 4} *b4* = {2, 8} *b5* = {5, 6}
costs *r1* = 8; *r2* = 6; *r3* = 5; *r4* =2; *r5* = 2

Bucket-elimination for combinatorial auction

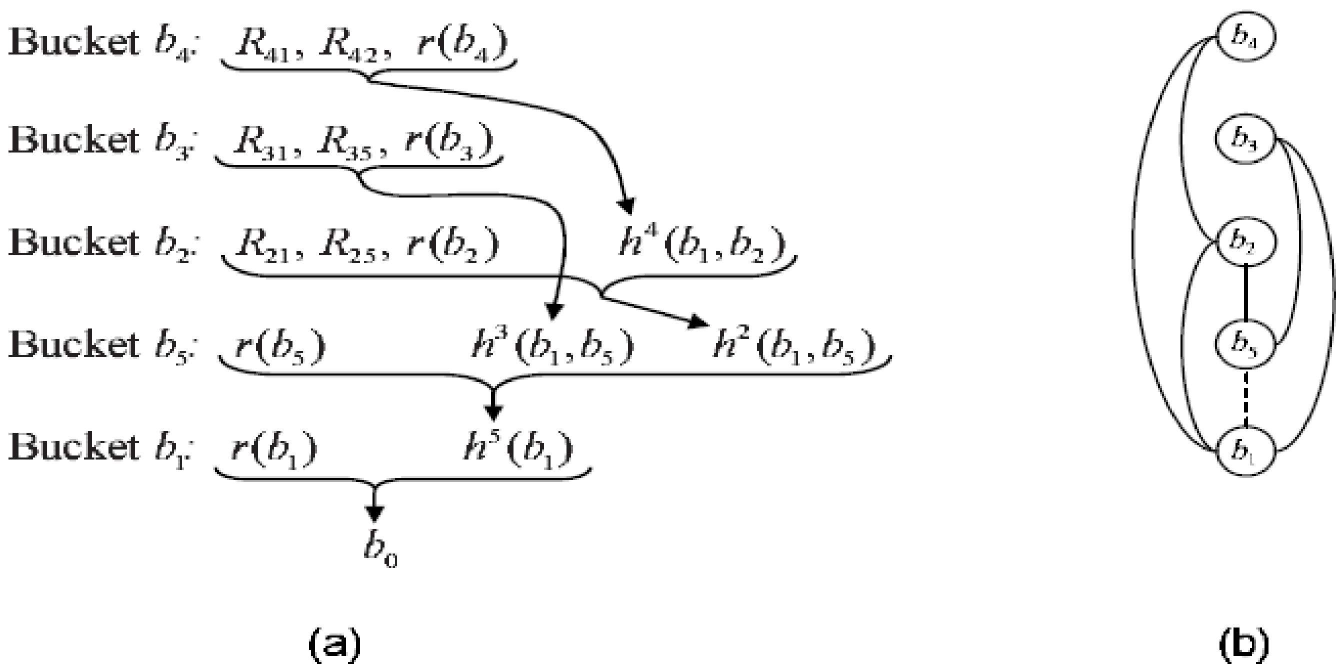


Figure 13.8: Schematic execution of elim-opt on the auction problem

Bucket-elimination for counting

Algorithm elim-count
Input: A constraint network $\mathcal{R} = (X, D, C)$, ordering d.
Output: Augmented output buckets including the intermediate count functions and The number of solutions.
1. **Initialize:** Partition C (0-1 cost functions) into ordered buckets $bucket_1, \ldots, bucket_n$, We denote a function in a bucket N_i, and its scope S_i.)
2. **Backward:** For $p \leftarrow n$ downto 1, do
   Generate the function N^p: $N^p = \sum_{X_p} \prod_{N_i \in bucket_p} N_i$.
   Add N^p to the bucket of the latest variable in $\bigcup_{i=1}^{j} S_i - \{X_p\}$.
3. **Return** the number of solutions, N^1 and the set of output buckets with the original and computed functions.

Figure 13.9: Algorithm *elim-count*

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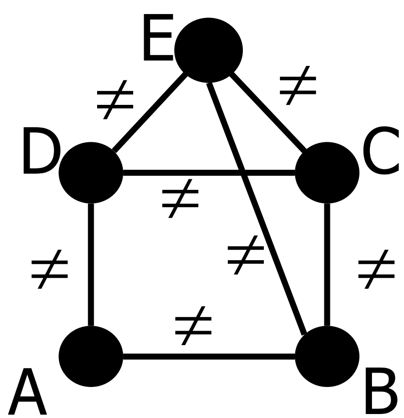
Directional i-consistency



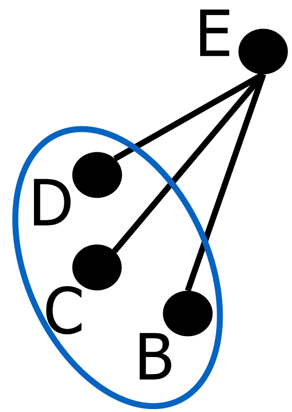
E : $E \neq D, E \neq C, E \neq B$

D : $D \neq C, D \neq A$

C : $C \neq B$

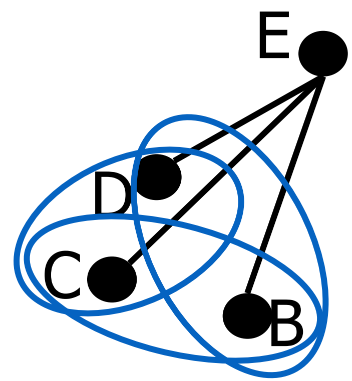
B : $A \neq B$

A :



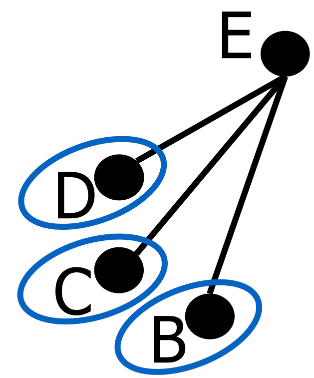
Adaptive

R_{DCB}



d-path

R_{DC}, R_{DB}
R_{CB}



d-arc

R_D
R_C
R_D

Mini-Bucket Elimination (MAP/MPE)

Split a bucket into mini-buckets —> bound complexity

bucket (X) =

$$\left\{ \underbrace{f_1,\ f_2,\ \dots\ f_r,\ f_{r+1},\ \dots\ f_n}_{} \right\}$$

$$\lambda_X(\cdot) = \max_x \prod_{i=1}^{n} f_i(x, \dots)$$

$$\left\{ f_1,\ \dots\ f_r \right\} \qquad \left\{ f_{r+1},\ \dots\ f_n \right\}$$

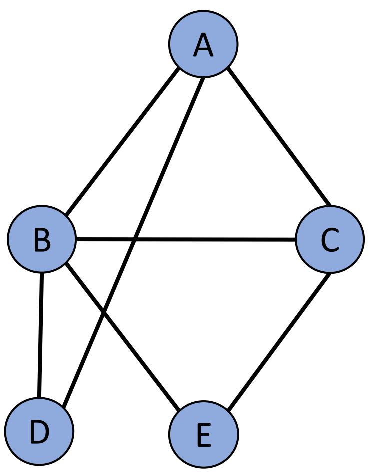
$$\lambda_{X,1}(\cdot) = \max_x \prod_{i=1}^{r} f_i(x, \dots) \qquad \lambda_{X,2}(\cdot) = \max_x \prod_{i=r+1}^{n} f_i(x, \dots)$$

$$\lambda_X(\cdot) \ \leq \ \lambda_{X,1}(\cdot)\ \lambda_{X,2}(\cdot)$$

Exponential complexity decrease: $O(e^n) \longrightarrow O(e^r) + O(e^{n-r})$

Mini-Bucket Elimination (MAP)

mini-buckets [Dechter & Rish 2003]



bucket B: $\underbrace{f(a,b)\ f(b,c)}\quad \underbrace{f(b,d)\ f(b,e)}$

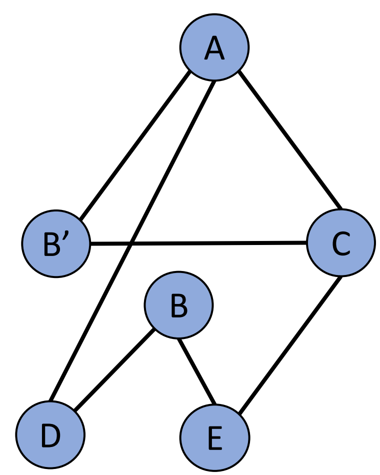
bucket C: $\underbrace{f(c,a)\ f(c,e)\ \lambda_{B\to C}(a,c)}$

bucket D: $\underbrace{f(a,d)\ \ \lambda_{B\to D}(d,e)}$

bucket E: $\underbrace{\lambda_{C\to E}(a,e)\ \lambda_{E\to E}(a,e)}$

bucket A: $\underbrace{f(a)\ \lambda_{E\to A}(a)}$

U = upper bound



$$\lambda_{B\to C}(a,c) = \max_b f(a,b)\ f(b,c)$$

$$\lambda_{B\to D}(d,e) = \max_b f(b,d)\ f(b,e)$$

$$\lambda_{C\to E}(a,e) = \max_c \ldots$$

$$\mathit{In\ our\ case}, \mathit{min} \sum$$

Mini-Bucket Decoding (MAP)

$$\mathbf{b}^* = \arg\max_b \; f(a^*, b) \cdot f(b, c^*) \cdot f(b, d^*) \cdot f(b, e^*)$$

$$\mathbf{c}^* = \arg\max_c \; f(c, a^*) \cdot f(c, e^*) \cdot \lambda_{B\to C}(a^*, c)$$

$$\mathbf{d}^* = \arg\max_d \; f(a^*, d) \cdot \lambda_{B\to D}(d, e^*)$$

$$\mathbf{e}^* = \arg\max_e \; \lambda_{C\to E}(a^*, e) \cdot \lambda_{D\to E}(a^*, e)$$

$$\mathbf{a}^* = \arg\max_a \; f(a) \cdot \lambda_{E\to A}(a)$$

Greedy configuration = lower bound

mini-buckets

B: $f(a,b)\ f(b,c)$ $\quad$ $f(b,d)\ f(b,e)$

C: $f(c,a)\ f(c,e)\ \lambda_{B\to C}(a,c)$

D: $f(a,d)\ \lambda_{B\to D}(d,e)$

E: $\lambda_{C\to E}(a,e)\ \lambda_{E\to E}(a,e)$

A: $f(a)\ \lambda_{E\to A}(a)$

U = upper bound

MBE vs BE

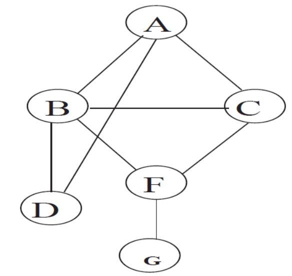
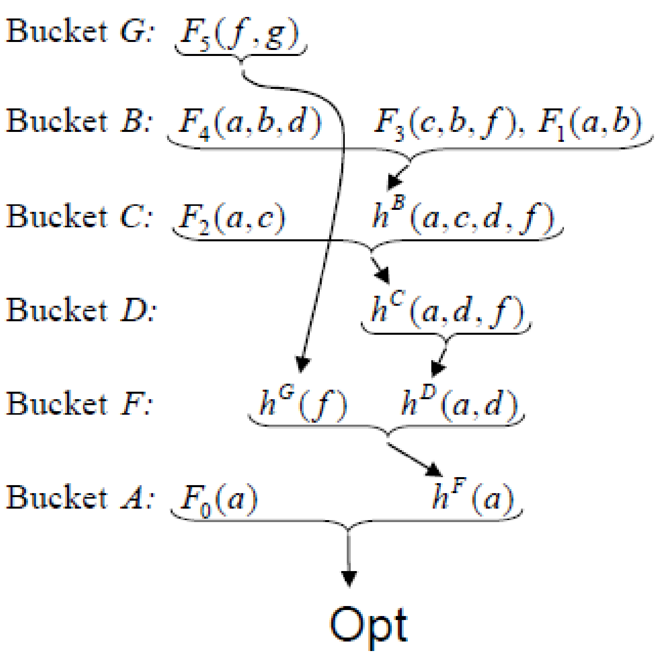
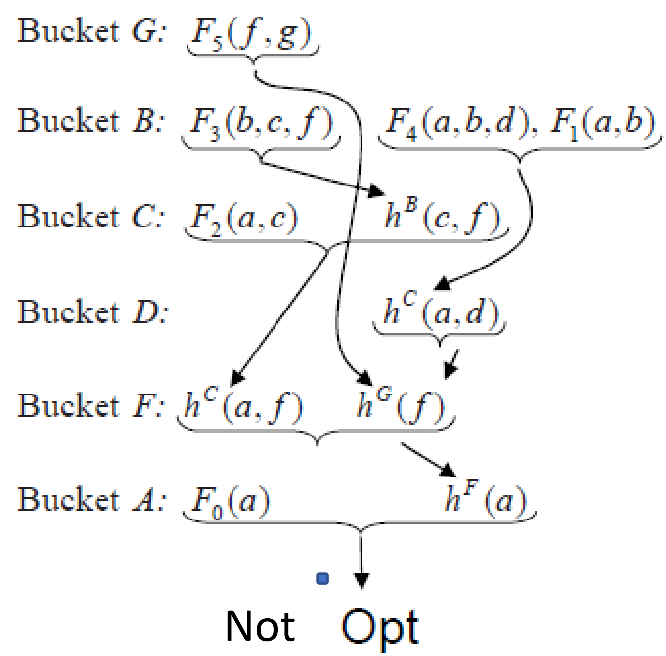


Figure 13.1: The cost graph of the cost function: $C(a,b,c,d,f,g) = F_0(a) + F_1(a,b) + F_2(a,c) + F_3(b,c,f) + F_4(a,b,d) + F_5(f,g)$



(a) A trace of *elim-opt*



(b) A trace of *mbe-opt(3)*

Bucket-elimination for combinatorial auction example

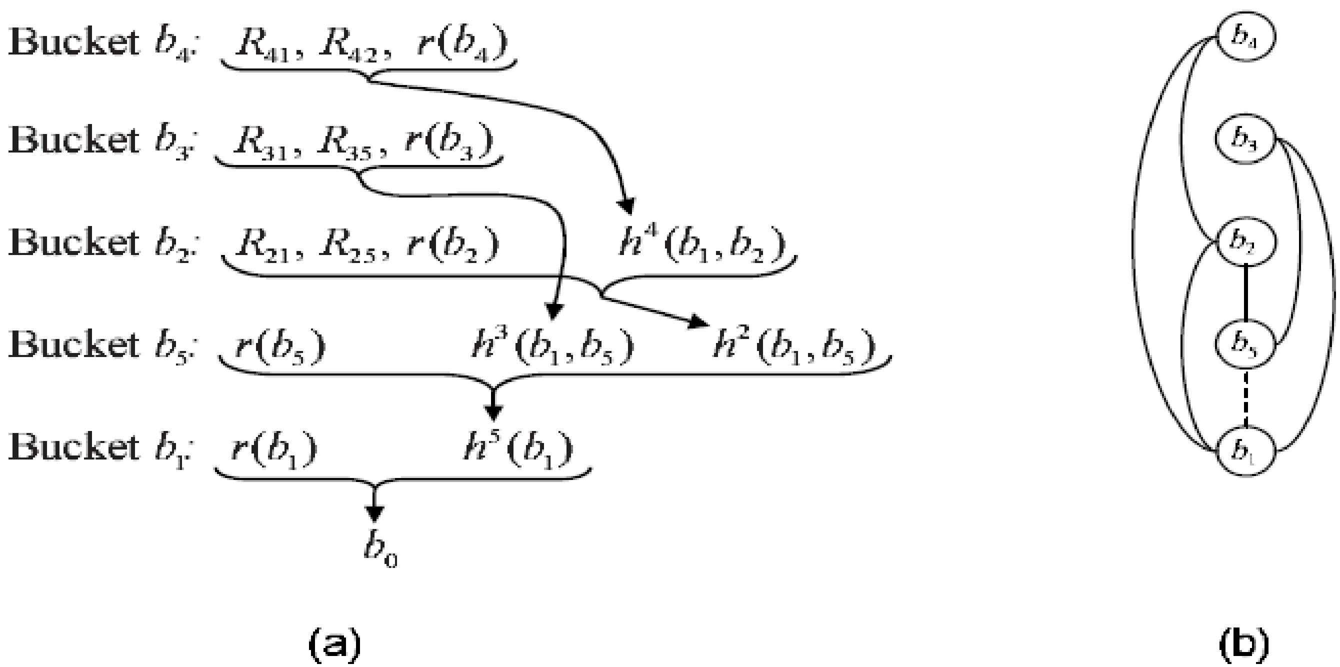


Figure 13.8: Schematic execution of elim-opt on the auction problem

Example mbe-opt

Example 13.4.3 Let us apply algorithm mbe-opt(2)to the auction problem along the ordering $d = b_1, b_5, b_2, b_3, b_4$. Figure 13.12 shows the resulting mini-buckets; square brackets denote the choice for partitioning.

We start with processing bucket b_4. A possible partitioning places the constraint R_{41} in one mini-bucket and the rest in the other. We compute a constraint $R^4(b_1) = \pi_{b_1} R_{41}$, which is the universal constraint so it need not be recorded. In the second mini-bucket, we also compute $h^4(b_2) = \max_{\{(b_4|(b_4,b_2)\in R_{42}\}} r(b_4)$, yielding:

$$h^4(b_2) = \begin{cases} 0 & \text{if } b_2 = 1 \\ 2 & \text{if } b_2 = 0 \end{cases}$$

Processing the first mini-bucket of b_3, which includes only hard constraints, will also not effect the domain of b_1. Processing the second mini-bucket of b_3 by $h^3(b_5) = \max_{\{(b_3|(b_3,b_5)\in R_{35}\}} r(b_3)$ yields:

$$h^3(b_5) = \begin{cases} 0 & \text{if } b_5 = 1 \\ 5 & \text{if } b_5 = 0 \end{cases}$$

Processing the second mini-bucket of b_2 (the first mini-bucket includes a constraint whose projection is a universal constraint) by $h^2(b_5) = \max_{\{(b_2|(b_2,b_5)\in R_{25}\}}(r(b_2) + h^4(b_2))$, gives us:

$$h^2(b_5) = \begin{cases} 2 & \text{if } b_5 = 1 \\ 6 & \text{if } b_5 = 0 \end{cases}$$

Processing bucket b_5 (with full buckets now) the algorithm computes $h^5 = \max_{b_5}(r(b_5) + h^2(b_5) + h^3(b_5))$, yielding:

$$h^5 = 11$$

Finally, in the bucket of b_1 the algorithm computes $h^1 = \max_{b_1}(r(b_1) + h^5)$, yielding: $h^1 = \max\{0 + 11, 8 + 11\} = 19$.

The maximal upper-bound cost is therefore 19. To compute a maximizing tuple we select in bucket b_1 the value $b_1 = 1$, which maximizes $r(b_1) + h^5$. Given $b_1 = 1$, we choose $b_5 = 0$, which maximizes the functions in bucket b_5. Then, in bucket b_2, we can choose only $b_2 = 0$, due to the constraint R_{12}. Likewise, the only subsequent choices are $b_3 = 0$ and $b_4 = 0$. Therefore, the cost of the generated solution is 8, yielding the interval [8,19] bounding the maximal solution. □

Bucket b_4 : $[R_{41}], [R_{42}, r(b_4)]$
Bucket b_3 : $[R_{31}], [R_{35}, r(b_3)]$
Bucket b_2 : $[R_{21}], [R_{25}, r(b_2) \;||\; h^4(b_2)]$
Bucket b_5 : $[r(b_5) \;||\; h^3(b_5), h^2(b_5)]$
Bucket b_1 : $r(b_1) \;||\; h^5$
Yielding: opt: $h^1 = M'$

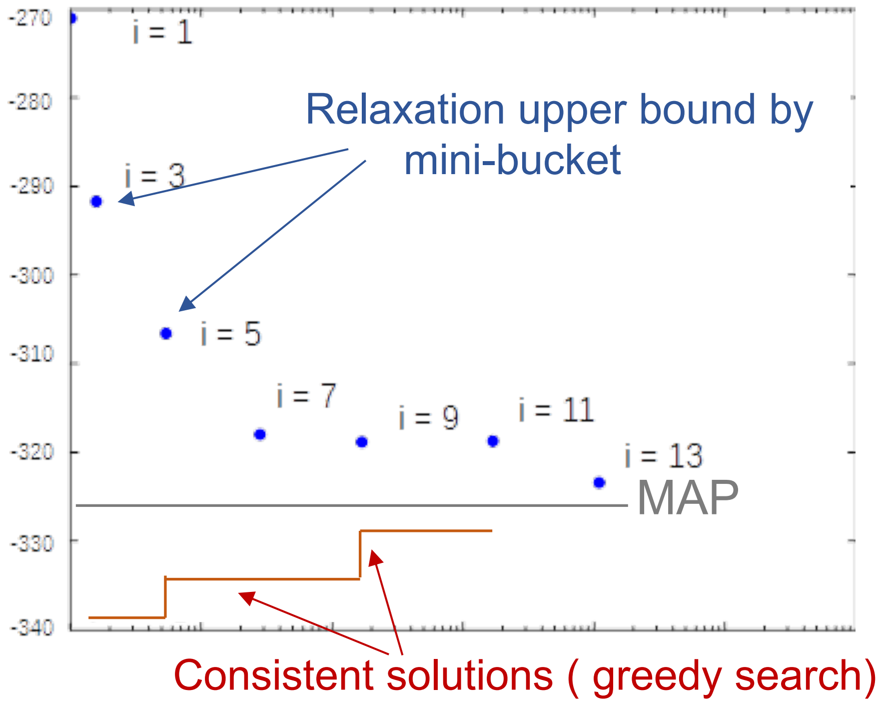
Algorithm mbe-opt

Algorithm mbe-opt(i)
Input: A cost network $\mathcal{C} = (X, D, C)$; an ordering d; parameter i.
Output: An upper bound on the optimal cost solution, a solution and a lower bound and the ordered augmented buckets.
1. **Initialize:** Partition the functions in C into $bucket_1, \ldots, bucket_n$, where $bucket_i$ contains all functions whose highest variable is x_i. Let $S_1, ..., S_j$ be the scopes of functions (new or old) in the processed bucket.
2. **Backward** For $p \leftarrow n$ down-to 1, do
• **If** variable x_p is instantiated ($x_p = a_p$), assign $x_p = a_p$ to each h_i and put each resulting function into its appropriate bucket.
• **Else,** for $h_1, h_2, ..., h_j$ in $bucket_p$, generate an (i)-partitioning, $Q' = \{Q_1, ..., Q_t\}$. For each $Q_l \in Q'$ containing $h_{l_1}, ...h_{l_t}$ generate function h^l, $h^l = \max_{x_p} \sum_{i=1}^{t} h_{l_i}$. Add h^l to the bucket of the largest-index variable in U_l, $U_l = \bigcup_{i=1}^{j} scope(h_{l_i}) - \{x_p\}$.
3. **Forward** For $i = 1$ to n do, given $a_1, ..., a_{p-1}$ choose a value a_p of x_p that maximizes the sum of all the functions in x_p's bucket.
4. **Return** the ordered set of augmented buckets, an assignment $\bar{a} = (a_1, ..., a_n)$, an interval bound (the value computed in $bucket_1$ and the cost $F(\bar{a})$).

Figure 13.9: Mini-Bucket Elimination Algorithm

Properties of Mini-Bucket Eliminaton

Bounding from above and below:



- **Complexity:** *O(r exp(i))* time and *O(exp(i)) space*.
- **Accuracy:** determined by Upper/Lower bound.
- As ***i*** increases, both accuracy and complexity increase.
- Possible use of mini-bucket approximations:
  - As anytime algorithms
  - As heuristics in search

Outline

- **Introduction**
  - Optimization tasks for graphical models
  - Solving optimization problems with inference and search
- **Inference**
  - Bucket elimination, dynamic programming
  - Mini-bucket elimination
- **Search**
  - Branch and bound and best-first
  - Lower-bounding heuristics
  - AND/OR search spaces

Branch and bound

procedure BRANCH-AND-BOUND
Input: A cost network $\mathcal{C} = (X, D, C_h, C_s)$, L current upper-bound, An upper-bound function f defined for every partial solution.
Output: Either an optimal (maximal) solution, or notification that the network is inconsistent.

$i \leftarrow 1$ (initialize variable counter)
$D'_i \leftarrow D_i$ (copy domain)
While $1 \leq i \leq n$
instantiate $x_i \leftarrow$ SELECTVALUE
If x_i is null (no value was returned)
$i \leftarrow i - 1$ (backtrack)
Else
$i \leftarrow i + 1$ (step forward)
$D'_i \leftarrow D_i$
Endwhile
If $i = 0$
Return "inconsistent"
Else
Compute $C = C(x_1, ...x_n)$, $U \leftarrow max\{C, L\}$
$i \leftarrow n - 1$
end procedure

procedure SELECTVALUE
If $i = 0$ return U as the solution value and the most recent assignment as solution.
While D'_i is not empty
select an element $a \in D'_i$ having max $f(\vec{a}_{i-1}, a)$
and remove a from D'_i
If $< x_i, a >$ is consistent with $\vec{a}_{i-1}$ and $f > L$, then
Return a (else prune a)

Endwhile
Return null (no consistent value)
end procedure

Figure 13.2: The Branch and Bound algorithm

Search tree for first-choice heuristic

$$f_{fc}(\vec{a}_i) = \sum_{F_j \in C} \max_{a_{i+1},\ldots a_n} F_j(\vec{a}_n)$$

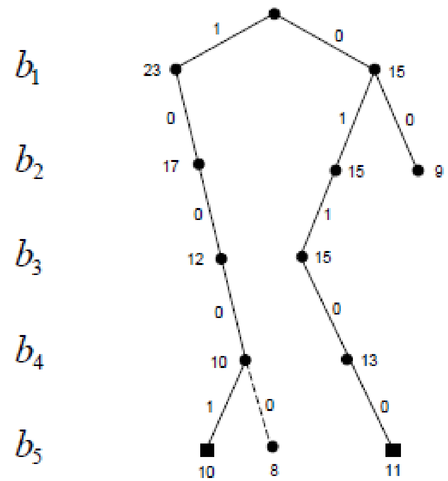
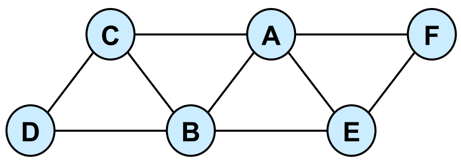


Figure 13.3: Branch and bound search space for the auction problem

Example 13.2.1 Consider the auction problem described in Example 13.1.2. Searching for a solution in the order $d = b_1, b_2, b_3, b_4, b_5$ yields the search space in Figure 13.3, traversed from left to right. The search space is highly constrained in this case. The evaluation bounding function at each node should *overestimates* its best extension for a solution. The first-choice bounding function for the root node (the empty assignment) is 23. The first solution encountered (selecting $< b_1, 1 >$ and $< b_5, 1 >$) has a cost of 10, which becomes the current global *lower bound*. The next solution encountered has the cost of 11 (when $< b_2, 1 >$ and $< b_3, 1 >$), while the rest of the variables are assigned 0). Subsequently, the partial assignment $(< b_1, 0 >, < b_2, 0 >)$ is explored. Since $f_{fc}(< b_1, 0 >, < b_2, 0 >) = 9$, this upper bound is lower than 11, and therefore search can be pruned. $\square$

The search space



A	B	f_1
0	0	**2**
0	1	**0**
1	0	**1**
1	1	**4**

A	C	f_2
0	0	**3**
0	1	**0**
1	0	**0**
1	1	**1**

A	E	f_3
0	0	**0**
0	1	**3**
1	0	**2**
1	1	**0**

A	F	f_4
0	0	**2**
0	1	**0**
1	0	**0**
1	1	**2**

B	C	f_5
0	0	**0**
0	1	**1**
1	0	**2**
1	1	**4**

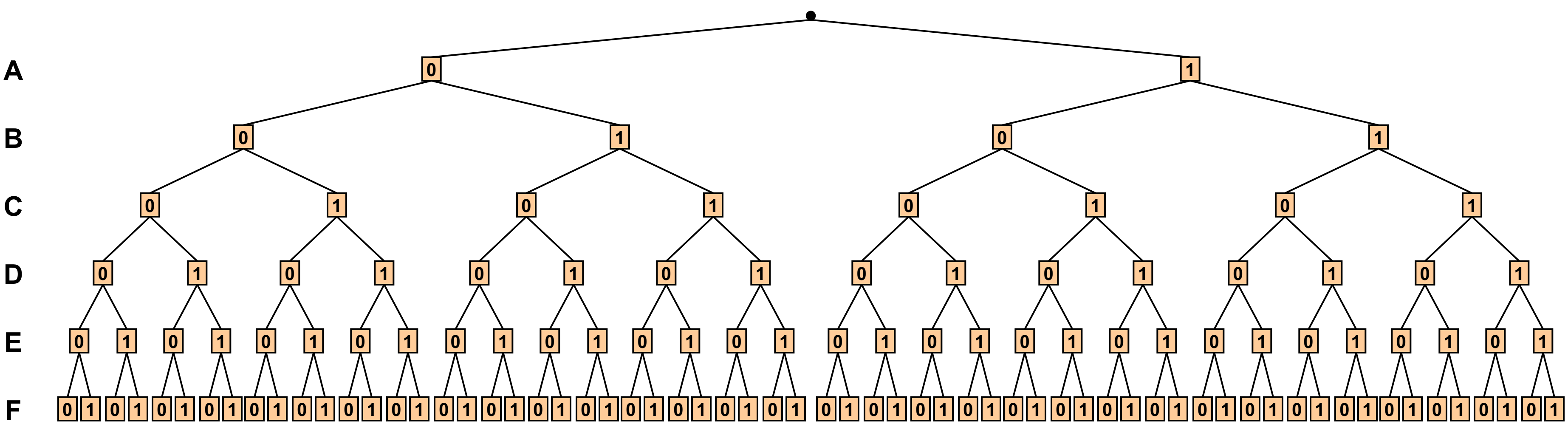
B	D	f_6
0	0	**4**
0	1	**2**
1	0	**1**
1	1	**0**

B	E	f_7
0	0	**3**
0	1	**2**
1	0	**1**
1	1	**0**

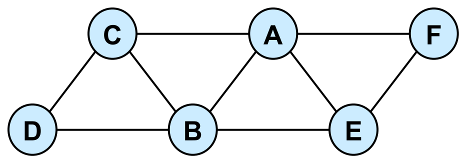
C	D	f_8
0	0	**1**
0	1	**4**
1	0	**0**
1	1	**0**

E	F	f_9
0	0	**1**
0	1	**0**
1	0	**0**
1	1	**2**

Objective function: $f(X) = \min_X \sum_{i=1}^{9} f_i(X)$



The search space



A	B	f_1
0	0	**2**
0	1	**0**
1	0	**1**
1	1	**4**

A	C	f_2
0	0	**3**
0	1	**0**
1	0	**0**
1	1	**1**

A	E	f_3
0	0	**0**
0	1	**3**
1	0	**2**
1	1	**0**

A	F	f_4
0	0	**2**
0	1	**0**
1	0	**0**
1	1	**2**

B	C	f_5
0	0	**0**
0	1	**1**
1	0	**2**
1	1	**4**

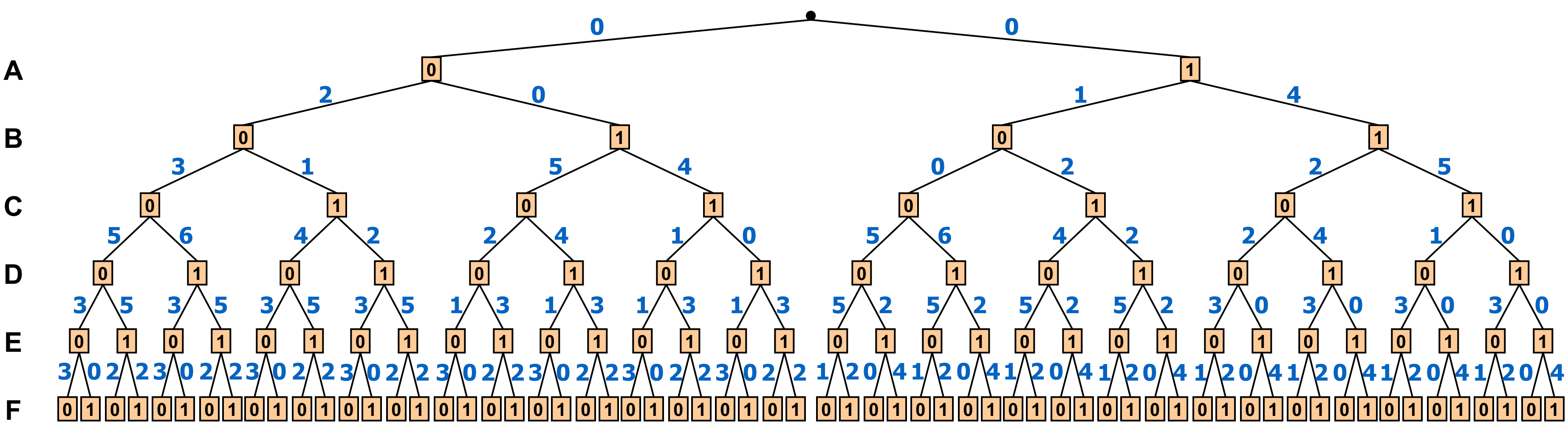
B	D	f_6
0	0	**4**
0	1	**2**
1	0	**1**
1	1	**0**

B	E	f_7
0	0	**3**
0	1	**2**
1	0	**1**
1	1	**0**

C	D	f_8
0	0	**1**
0	1	**4**
1	0	**0**
1	1	**0**

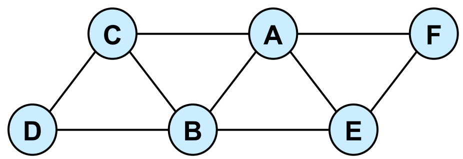
E	F	f_9
0	0	**1**
0	1	**0**
1	0	**0**
1	1	**2**

$$f(\mathrm{X}) = \min_X \sum_{i=1}^{9} f_i(\mathrm{X})$$



Arc-cost is calculated based on cost components.

The value function



$$f(X) = \min_X \sum_{i=1}^{9} f_i(X)$$

A	B	f_1
0	0	**2**
0	1	**0**
1	0	**1**
1	1	**4**

A	C	f_2
0	0	**3**
0	1	**0**
1	0	**0**
1	1	**1**

A	E	f_3
0	0	**0**
0	1	**3**
1	0	**2**
1	1	**0**

A	F	f_4
0	0	**2**
0	1	**0**
1	0	**0**
1	1	**2**

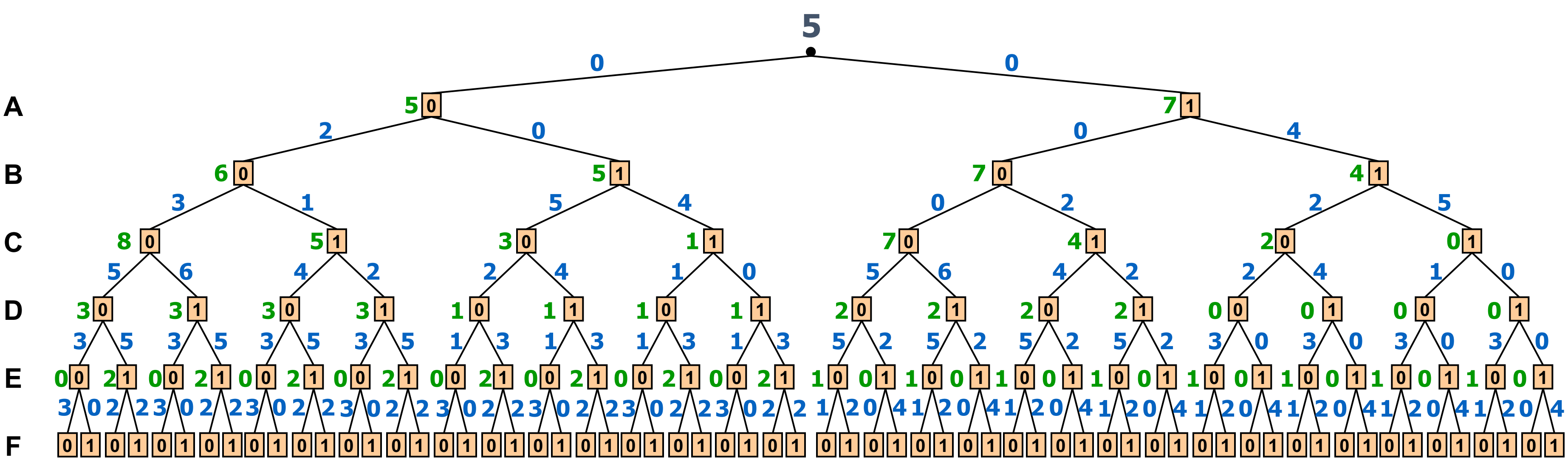
B	C	f_5
0	0	**0**
0	1	**1**
1	0	**2**
1	1	**4**

B	D	f_6
0	0	**4**
0	1	**2**
1	0	**1**
1	1	**0**

B	E	f_7
0	0	**3**
0	1	**2**
1	0	**1**
1	1	**0**

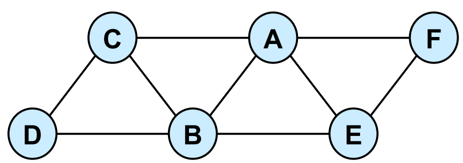
C	D	f_8
0	0	**1**
0	1	**4**
1	0	**0**
1	1	**0**

E	F	f_9
0	0	**1**
0	1	**0**
1	0	**0**
1	1	**2**



Value of node = minimal cost solution below it

An optimal solution



A	B	f_1
0	0	**2**
0	1	**0**
1	0	**1**
1	1	**4**

A	C	f_2
0	0	**3**
0	1	**0**
1	0	**0**
1	1	**1**

A	E	f_3
0	0	**0**
0	1	**3**
1	0	**2**
1	1	**0**

A	F	f_4
0	0	**2**
0	1	**0**
1	0	**0**
1	1	**2**

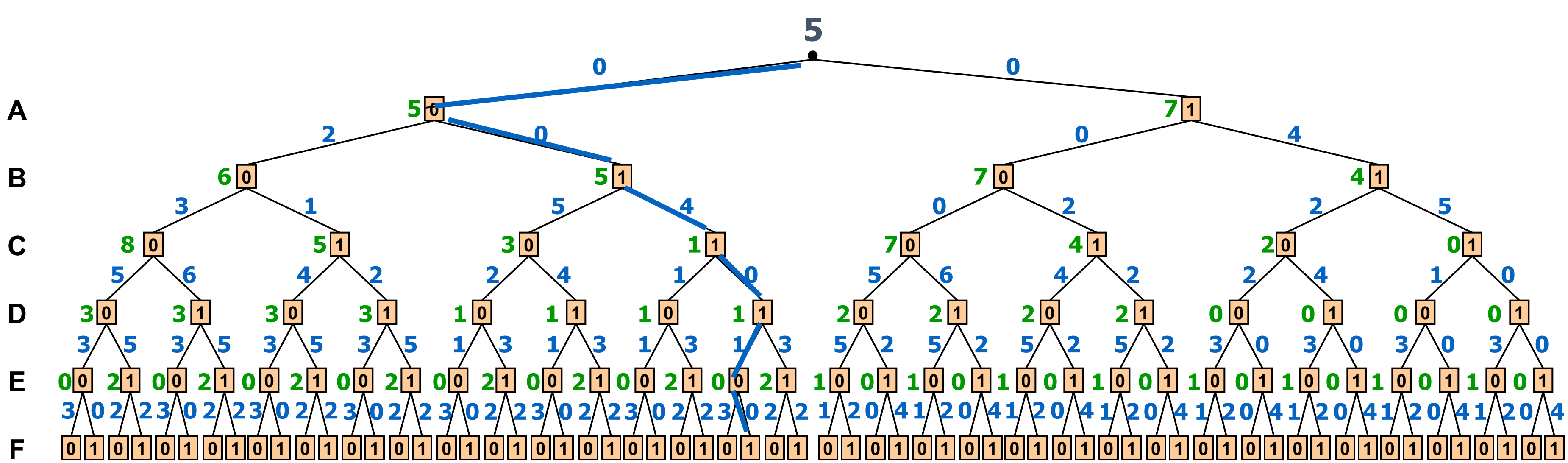
B	C	f_5
0	0	**0**
0	1	**1**
1	0	**2**
1	1	**4**

B	D	f_6
0	0	**4**
0	1	**2**
1	0	**1**
1	1	**0**

B	E	f_7
0	0	**3**
0	1	**2**
1	0	**1**
1	1	**0**

C	D	f_8
0	0	**1**
0	1	**4**
1	0	**0**
1	1	**0**

E	F	f_9
0	0	**1**
0	1	**0**
1	0	**0**
1	1	**2**



Value of node = minimal cost solution below it

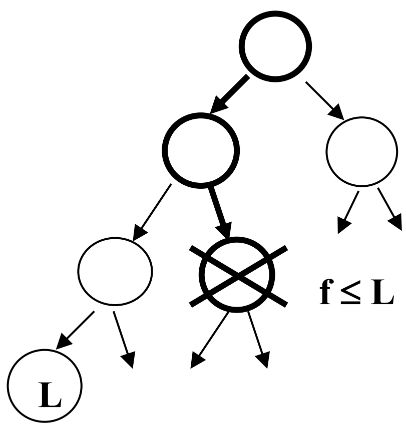
Basic heuristic search schemes

Heuristic function f(x) computes a lower bound on the best extension of x and can be used to guide a heuristic search algorithm. We focus on

1.Branch and Bound

Use heuristic function $f(x^p)$ to prune the depth-first search tree.

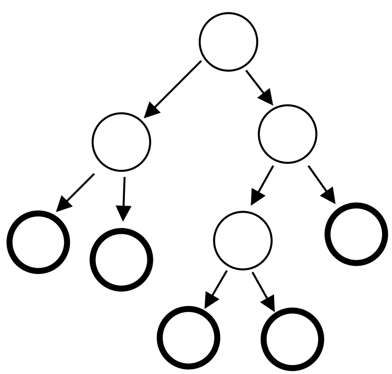
Linear space



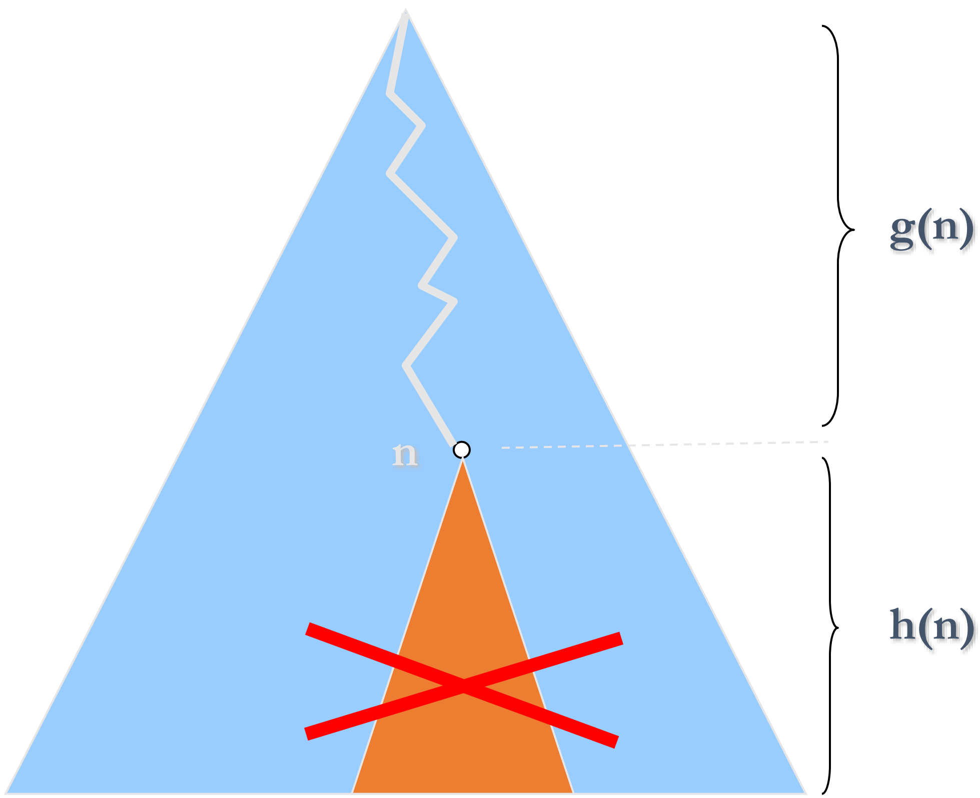
2.Best-First Search

Always expand the node with the highest heuristic value $f(x^p)$.

Needs lots of memory



Classic branch-and-bound



OR Search Tree

Upper Bound **UB**

Lower Bound **LB**

LB(n) = g(n) + h(n)

Prune if LB(n) ≥ UB

How to Generate Heuristics

- The principle of relaxed models
  - Linear optimization for integer programs
  - Mini-bucket elimination
  - Bounded directional consistency ideas

Generating Heuristics for Graphical Models

Given a cost function:

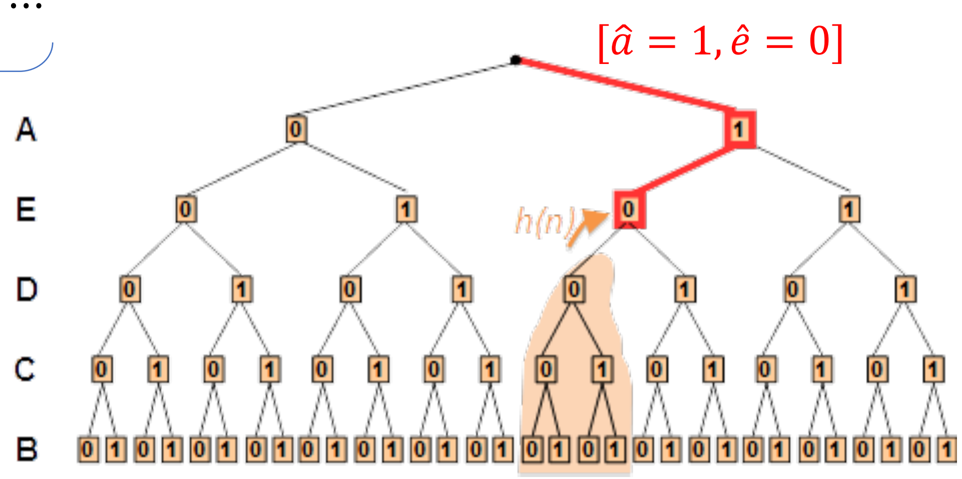
$$f(a, \dots, e) = f(a) + f(a,b) + f(a,c) + f(a,d) + f(b,c) + f(b,d) + f(b,e) + f(c,e)$$

define an evaluation function over a partial assignment as the cost of its best extension:

$$f^*(\hat{a}, \hat{e}, D) = \min_{b,c} F(\hat{a}, b, c, D, \hat{e})$$

$$= \underbrace{f(\hat{a})} + \underbrace{\min_{b,c} f(\hat{a}, b) + f(\hat{a}, c) + \cdots}$$

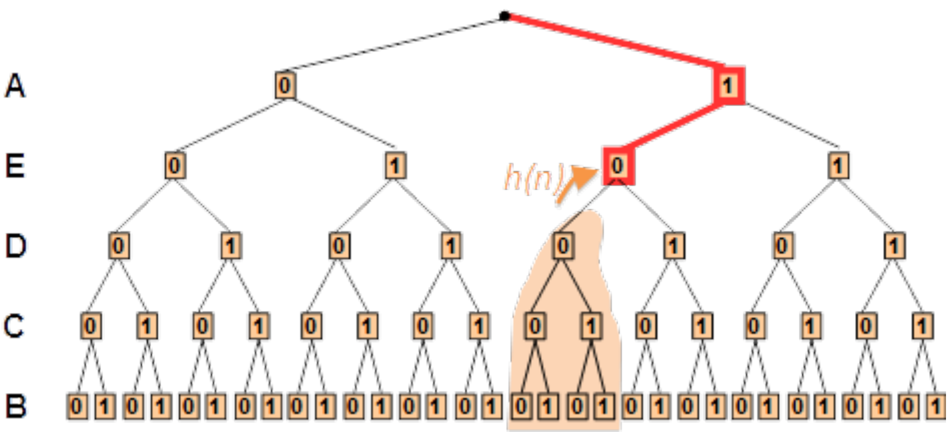
$$= g(\hat{a}, \hat{e}, D) + h^*(\hat{a}, \hat{e}, D)$$



[Kask and Dechter, 2001]

Static Mini-Bucket Heuristics

Given a partial assignment, $[\hat{a} = 1, \hat{e} = 0]$
mini-bucket gives an admissible heuristic:



mini-buckets

$\min_B \sum f(\cdot)$

bucket B: $\underbrace{f(a,b)\ f(b,c)}$ $\underbrace{f(b,d)\ f(b,e)}$

bucket C: $\lambda_{B\to C}(a,c)\ f(c,a)\ f(c,e)$

bucket D: $f(a,d)\ \lambda_{B\to D}(d,e)$

bucket E: $\lambda_{C\to E}(a,e)\ \lambda_{D\to E}(a,e)$

bucket A: $f(a)\quad \lambda_{E\to A}(a)$

L = lower bound

cost to go:

$$\tilde{h}(\hat{a},\hat{e},D) = \lambda_{C\to E}(\hat{a},\hat{e}) + f(\hat{a},D) + \lambda_{B\to D}(D,\hat{e})$$

(admissible: $\tilde{h}(\hat{a},\hat{e},D) \le h^*(\hat{a},\hat{e},D)$)

cost so far:

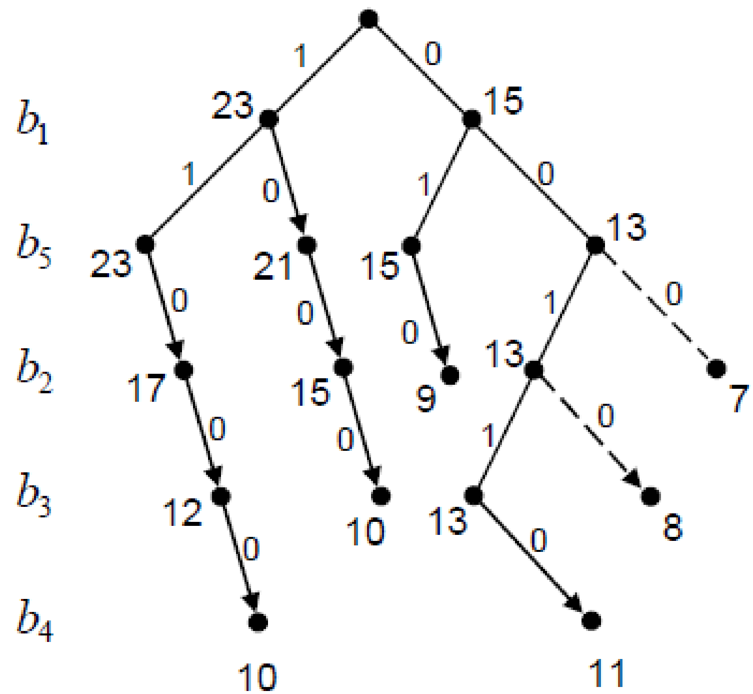
$$g(\hat{a},\hat{e},D) = f(A=\hat{a})$$

BBMB(i) with mini-bucket heuristics

$b1 = \{1; 2; 3; 4\}$; $b2 = \{2; 3; 6\}$;
$b3 = \{1; 5; 4\}$ $b4 = \{2; 8\}$ $b5 = \{5; 6\}$
costs $r1 = 8$; $r2 = 6$; $r3 = 5$; $r4 = 2$; $r5 = 2$

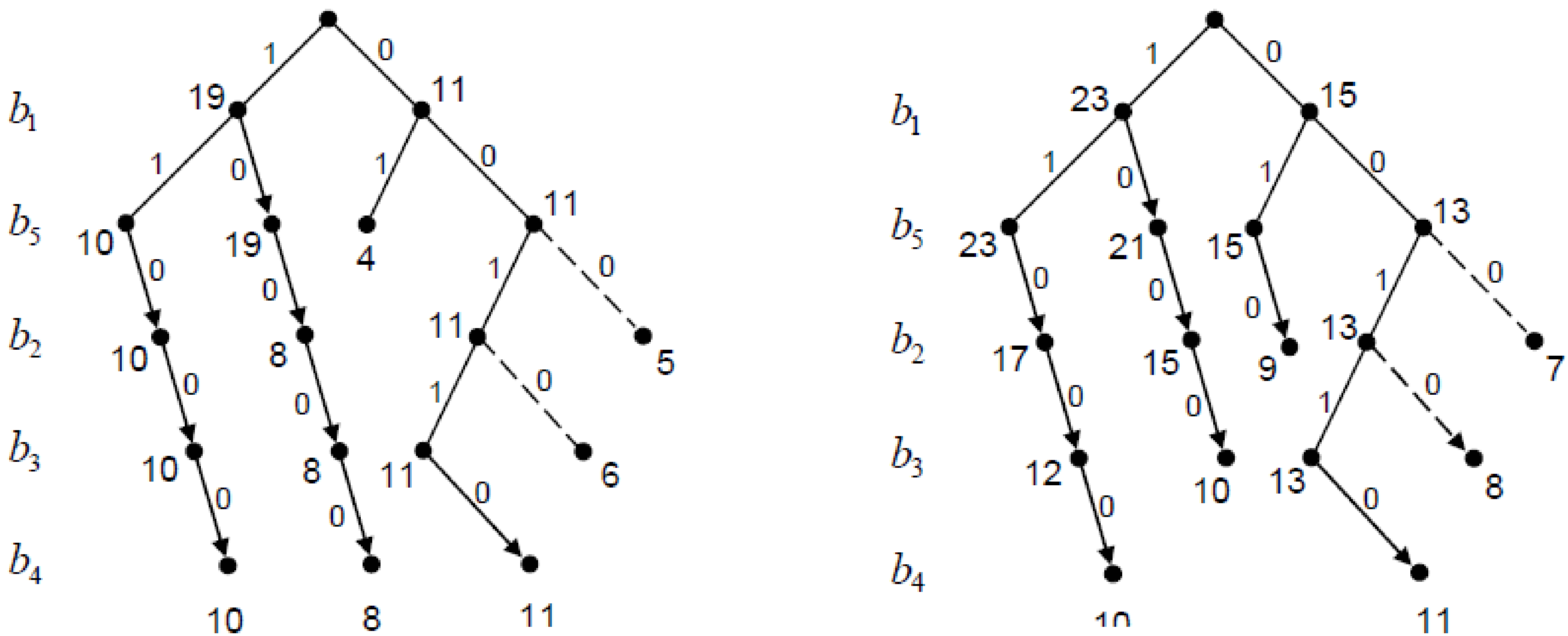
Let us now apply BnB with f_{mb}, which is the bounding function extracted from mbe-opt(2). Based on the functions in the augmented bucket of b_1 produced by mbe-opt(2), $f_{mb}(b_1 = 0) = r(b_1) + h^5 = 8 + 11 = 19$ while $f_{mb}(b_1 = 1) = 0 + 11 = 11$. Consequently, $b_1 = 1$ is chosen. We next evaluate $f_{mb}(b_1 = 1, b_5) = 8 + h^3(b_5) + h^2(b_5)$, yielding $f_{mb}(b_5 = 0) = 19$ and $f_{mb}(b_5 = 1) = 10$. Consequently, $b_5 = 0$ is chosen. The path is now deterministic, allowing the only choices: $b_2 = b_3 = b_4 = 0$. We end up with a solution having a cost of 8, which becomes the first global lower bound $L = 8$. BnB backtracks. The path is deterministic, dictating the choices $(b_2 = b_3 = b_4 = 0)$ whose bounding cost equals 10, yielding a solution with cost 10 $(b_1 = 1, b_5 = 1, b_2 = 0, b_3 = 0, b_4 = 0,)$. The global lower bound L is updated to 10. The algorithm backtracks to the next choice point, which is $(b_1 = 0)$, whose bounding cost is 11. Next, for b_5, $f_{mb}(b_1 = 0, b_5) = 0 + r(b_5) + h^2(b_5) + h^3(b_5)$, yielding $f_{mb}(b_5 = 1) = 4$, which can be immediately pruned (less than 10), and $f_{mb}(b_5 = 1) = 11$. We select $b_5 = 0$. Subsequently, choosing a value for b_2 we get: $f_{mb}(b_1 = 0, b_5 = 0, b_2) = r(b_2) + h^4(b_2) + h^3(b_5)$ (note that $h^2(b_5)$ is not included since it is created in bucket b_2). This yields $f_{mb}(b_2 = 1) = 11$, while $f_{mb}(b_2 = 0) = 5$, which is pruned. The next choice for b_3 is determined using the bounding function $f_{mb}(b_1 = 0, b_5 = 0, b_2 = 1, b_3) = 6 + r(b_3) + h^4(b_2 = 1)$, yielding for $b_3 = 1$ the bound 11, while for $b_3 = 0$ the bound 6, which will be pruned. $b_3 = 1$ is selected. Subsequently, only $b_4 = 0$ is feasible and we get the best cost solution of value 11. The global lower bound, L is updated to 11 and BnB will lead to only pruned choices.

We see in this example that the performance of BnB using these two bounding func-



Bucket b_4 : $[R_{41}], [R_{42}, r(b_4)]$
Bucket b_3 : $[R_{31}], [R_{35}, r(b_3)]$
Bucket b_2 : $[R_{21}], [R_{25}, r(b_2) \ || \ h^4(b_2)]$
Bucket b_5 : $[r(b_5) \ || \ h^3(b_5), h^2(b_5)]$
Bucket b_1 : $r(b_1) \ || \ h^5$
Yielding: opt: $h^1 = M'$

BnB with first-cut (b) and mini-bucket (a) heuristics (BBMB(i))



mb heuristic (a)

Bucket b_4 : $[R_{41}], [R_{42}, r(b_4)]$
Bucket b_3 : $[R_{31}], [R_{35}, r(b_3)]$
Bucket b_2 : $[R_{21}], [R_{25}, r(b_2) \;\; || \;\; h^4(b_2)]$
Bucket b_5 : $[r(b_5) \;\; || \;\; h^3(b_5), h^2(b_5)]$
Bucket b_1 : $r(b_1) \;\; || \; h^5$
Yielding: opt: $h^1 = M'$

(b) First cut heuristic

Heuristics properties

- MBE Heuristic is monotone, admissible
- Retrieved in linear time
- IMPORTANT:
  - Heuristic strength can vary by MBE(i).
  - Higher i-bound $\Rightarrow$ more pre-processing $\Rightarrow$ stronger heuristic $\Rightarrow$ less search.
- Allows controlled trade-off between preprocessing and search

Experimental methodology

Algorithms

- BBMB(i) – Branch and Bound with MB(i)
- BBFB(i) - Best-First with MB(i)
- MBE(i)

Test networks:

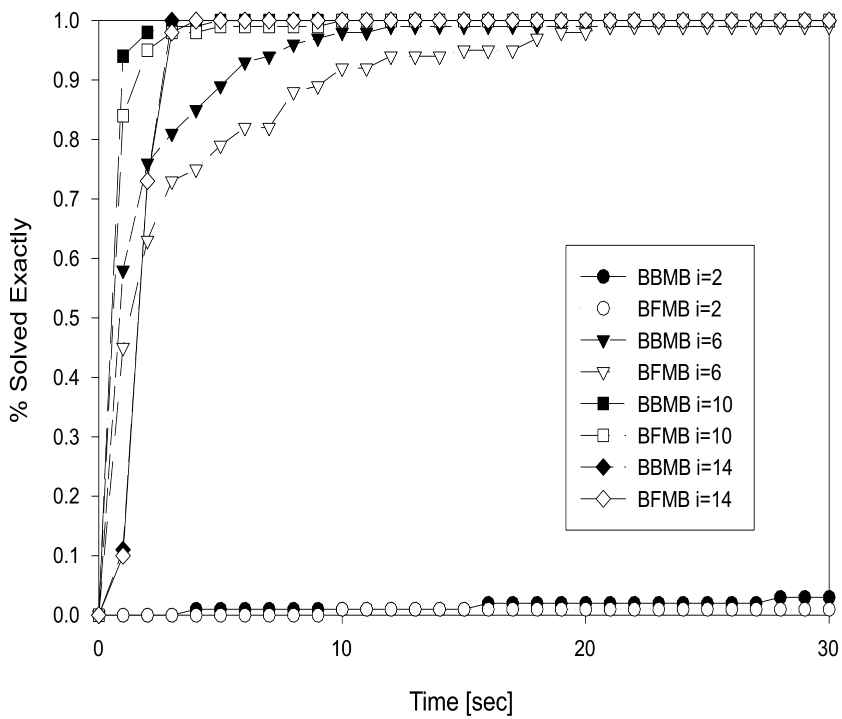
- Random Coding (Bayesian)
- CPCS (Bayesian)
- Random (CSP)

Measures of performance

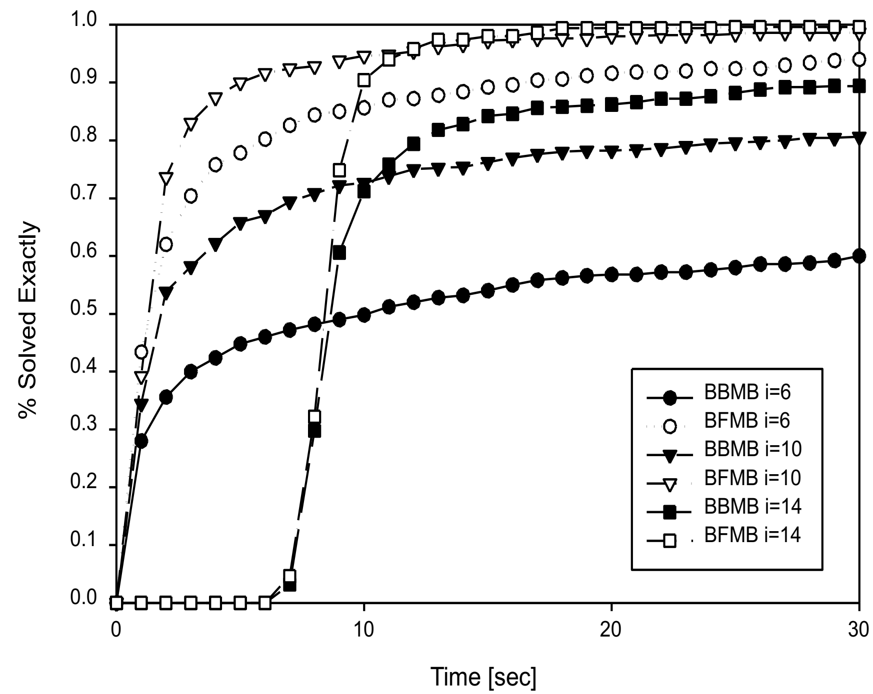
- Compare accuracy given a fixed amount of time - how close is the cost found to the optimal solution
- Compare trade-off performance as a function of time

Empirical evaluation of mini-bucket heuristics, Bayesian networks, coding

Random Coding, K=100, noise=0.28



Random Coding, K=100, noise=0.32



Max-CSP experiments

(Kask and Dechter, 2000)

T	MBE BBMB BFMB i=2 #/time	MBE BBMB BFMB i=4 #/time	MBE BBMB BFMB i=6 #/time	MBE BBMB BFMB i=8 #/time	MBE BBMB BFMB i=10 #/time	MBE BBMB BFMB i=12 #/time	PFC-MRDAC #/time
N=100, K=3, C=200. Time bound 1 hr. Avg w^*=21. Sparse network.							
1	70/0.03 90/12.5 80/0.03	90/0.06 **100/0.07** **100/0.07**	100/0.32 100/0.33 100/0.33	100/2.15 100/2.16 100/2.15	100/15.1 100/15.1 100/15.1	100/116 100/116 100/116	100/0.08
2	0/- 0/- 0/-	0/- 0/- 0/-	4/0.35 96/644 56/131	20/2.28 **92/41** 88/170	20/15.6 96/69 92/135	24/123 100/125 100/130	100/757
3	0/- 0/- 0/-	0/- 0/- 0/-	0/- 100/996 16/597	0/- 100/326 60/462	4/14.4 **100/94.6** 88/344	4/114 100/190 84/216	100/2879
4	0/- 0/- 0/-	0/- 0/- 0/-	0/- 52/2228 4/2934	0/- 88/1042 8/540	4/14.9 92/396 28/365	8/120 **100/283** 60/866	100/7320

Dynamic mbe heuristics

- Rather than pre-compiling, the mini-bucket heuristics can be generated during search
- D*ynamic mini-bucket heuristics* use the Mini-Bucket algorithm to produce a bound for any node in the search space

(a partial assignment, along the given variable ordering)

Branch and bound w/ mini-buckets

- BB with static Mini-Bucket Heuristics (s-BBMB)
  - Heuristic information is pre-compiled before search. Static variable ordering, prunes current variable
- BB with dynamic Mini-Bucket Heuristics (d-BBMB)
  - Heuristic information is assembled during search. Static variable ordering, prunes current variable
- BB with dynamic Mini-Bucket-Tree Heuristics (BBBT)
  - Heuristic information is assembled during search. Dynamic variable ordering, prunes all future variables

Empirical evaluation

- Algorithms:
  - Complete
    - BBBT
    - BBMB
  - Incomplete
    - DLM
    - GLS
    - SLS
    - IJGP
    - IBP (coding)

- Measures:
  - Time
  - Accuracy (% exact)
  - #Backtracks
  - Bit Error Rate (coding)

- Benchmarks:
  - Coding networks
  - Bayesian Network Repository
  - Grid networks (N-by-N)
  - Random noisy-OR networks
  - Random networks

Real World Benchmarks

Network	# vars	avg. dom.	max dom.	BBBT/ BBMB/ IJGP i=2 %[time]	BBBT/ BBMB/ IJGP i=4 %[time]	BBBT/ BBMB/ IJGP i=6 %[time]	BBBT/ BBMB/ IJGP i=8 %[time]	GLS % [time]	DLM % [time]	SLS % [time]
Mildew	35	17	100	**100[0.28]** 30[10.5] 90[3.59]	**100[0.56]** 95[0.18] 97[33.3]	- - -	- - -	15 [30.02]	0 [30.02]	90 [30.02]
Munin2	1003	5	21	95[1.65] 95[30.3] 95[2.44]	95[1.65] 95[30.5] 95[5.17]	95[2.32] 95[31.3] 95[64.9]	**100[1.97]** **100[1.84]** -	0 [30.01]	0 [30.01]	0 [30.01]
Pigs	441	3	3	**90[15.2]** 0[30.01] 80[0.31]	**100[3.73]** 60[4.85] 77[0.53]	**100[2.36]** 80[0.02] 80[1.43]	**100[0.58]** 95[0.04] 83[6.27]	10 [30.02]	0 [30.02]	0 [30.02]
CPCS360b	360	2	2	100[0.17] **100[0.04]** 100[10.6]	100[0.27] **100[0.03]** 100[10.5]	100[0.21] **100[0.03]** 100[9.82]	100[0.19] **100[0.03]** 100[8.59]	100 [30.02]	100 [30.02]	100 [30.02]

Average Accuracy and Time. 30 samples, 10 observations, 30 seconds

Empirical Results: Max-CSP

- **Random Binary Problems**: <N, K, C, T>
  - N: number of variables
  - K: domain size
  - C: number of constraints
  - T: Tightness

- **Task**: Max-CSP

BBBT(i) vs BBMB(i), N=100

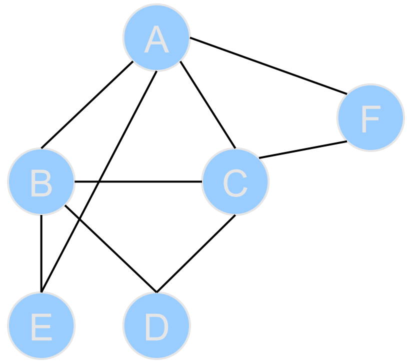
$N = 100,\ K = 5,\ C = 300.\ w^* = 33.9.$ 10 instances. time = 600sec.								
	BBMB						BBBT	PFC-MPRDAC
T	i=2	i=3	i=4	i=5	i=6	i=7	i=2	
	# solved time backtracks	# solved time backtracks	# solved time backtracks	# solved time backtracks	# solved time backtracks	# solved time backtracks	# solved time backtracks	# solved time backtracks
3	6 6 150K	6 6 150K	6 6 150K	6 5 115K	8 6.8 115K	8 15 8	10 7.73 60	10 0.03 750
5	2 36 980K	2 32 880K	2 24 650K	2 5.3 130K	3 38 870K	3 33 434K	10 14.3 114	10 0.06 1.5K
7	0	0	0	0	0	0	10 29 331	6 267 1.6M

BBBT(i) vs. BBMB(i).

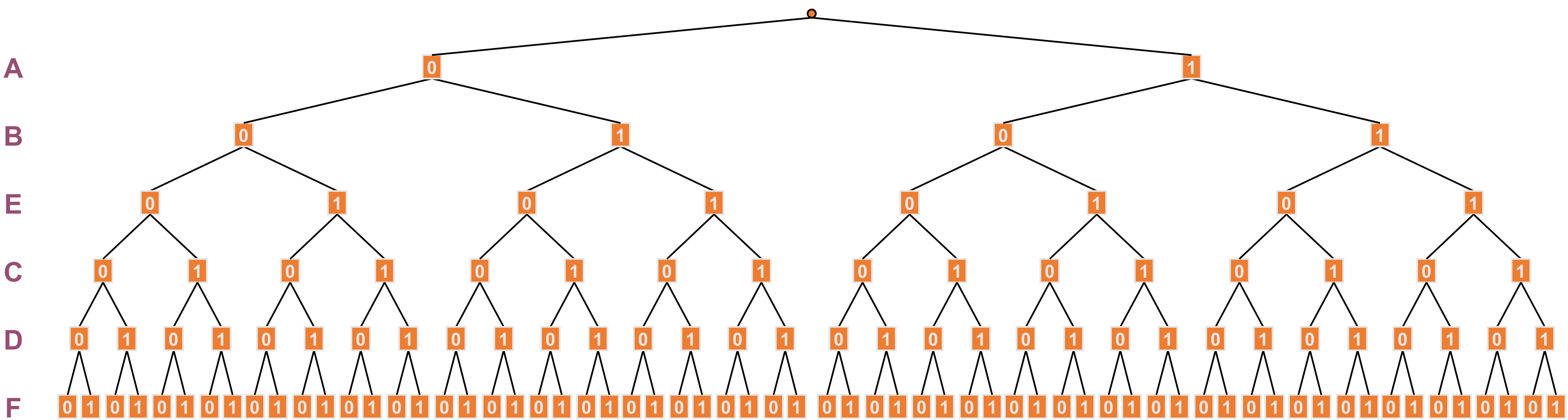
Outline

- **Introduction**
  - Optimization tasks for graphical models
  - Solving optimization problems with inference and search
- **Inference**
  - Bucket elimination, dynamic programming
  - Mini-bucket elimination
- **Search**
  - Branch and bound and best-first
  - Lower-bounding heuristics
  - AND/OR search spaces

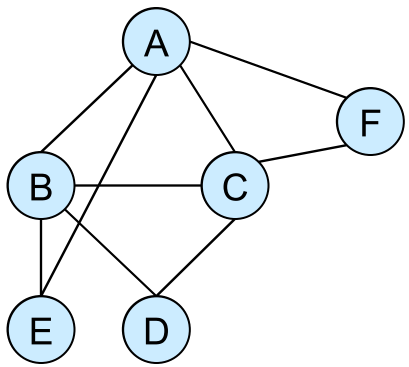
Classic OR search space



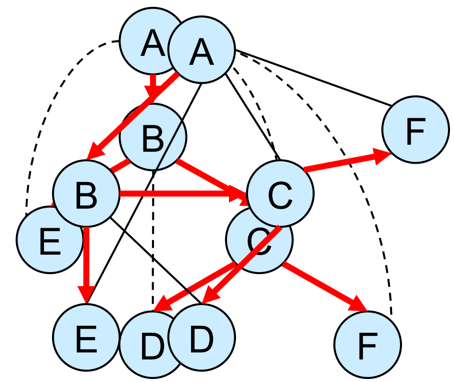
Ordering: A B E C D F



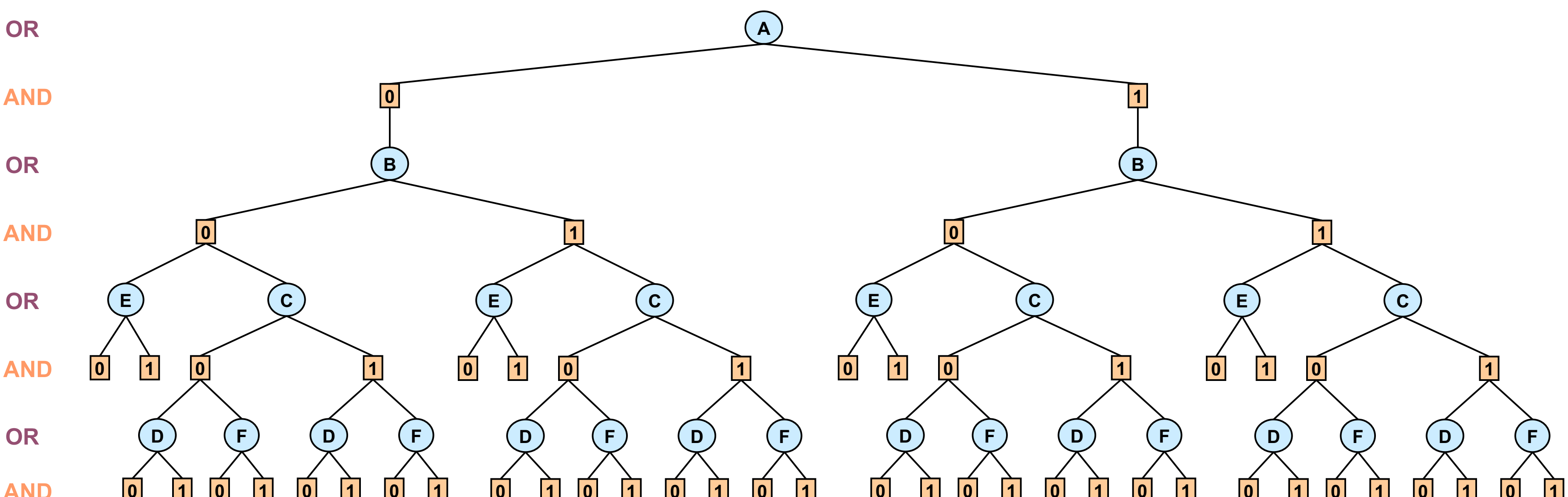
AND/OR search space



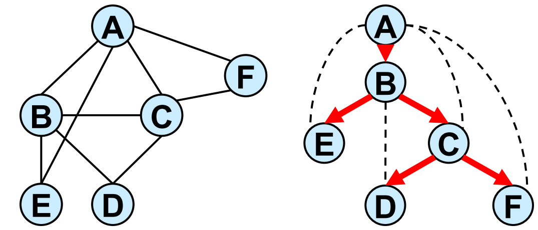
Primal graph

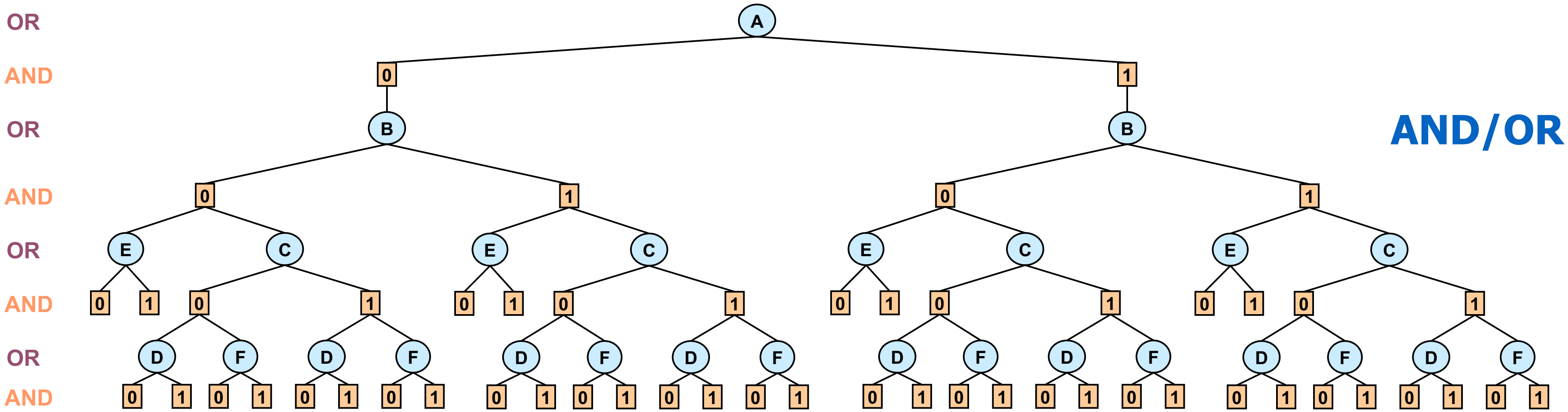


DFS tree

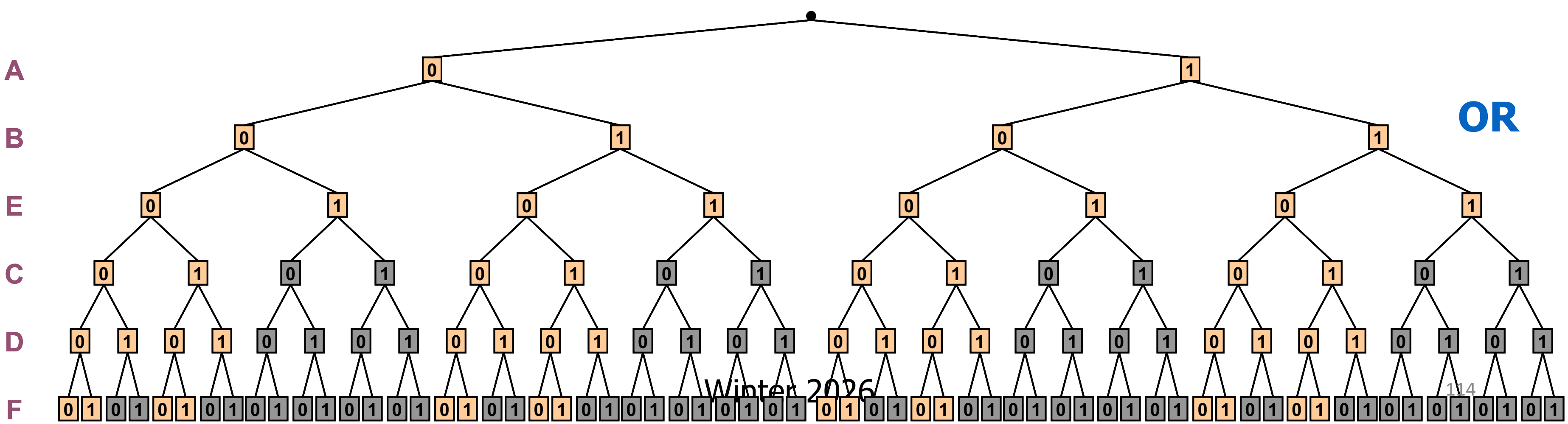


AND/OR vs. OR





AND/OR size: exp(4), OR size exp(6)



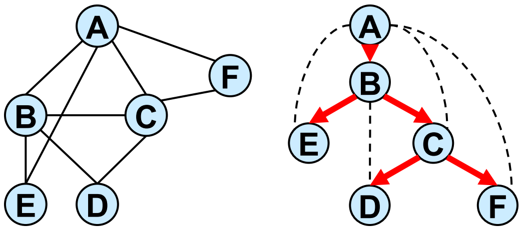
OR space vs. AND/OR space

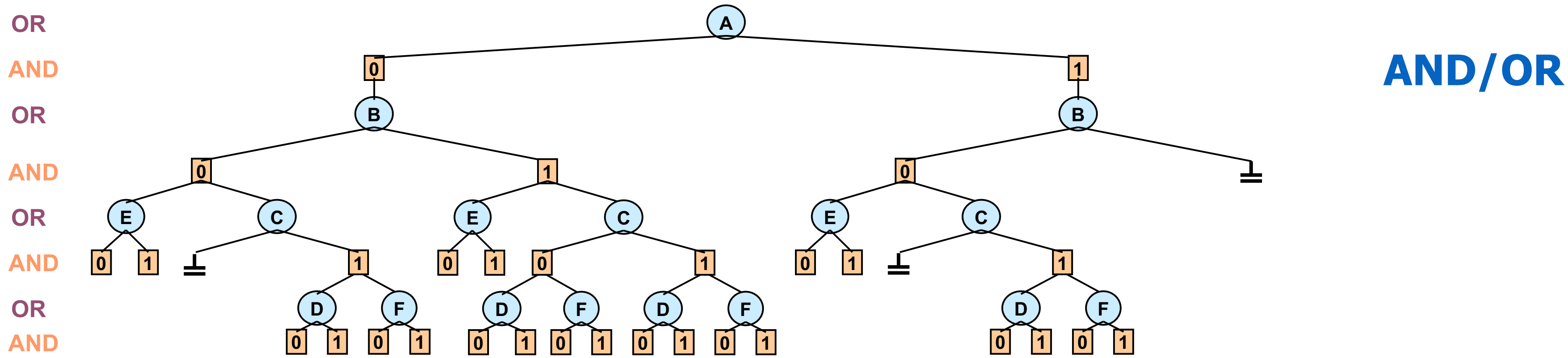
width	height	OR space			AND/OR space		
		Time (sec.)	Nodes	Backtracks	Time (sec.)	AND nodes	OR nodes
5	10	3.154	2,097,150	1,048,575	0.03	10,494	5,247
4	9	3.135	2,097,150	1,048,575	0.01	5,102	2,551
5	10	3.124	2,097,150	1,048,575	0.03	8,926	4,463
4	10	3.125	2,097,150	1,048,575	0.02	7,806	3,903
5	13	3.104	2,097,150	1,048,575	0.1	36,510	18,255

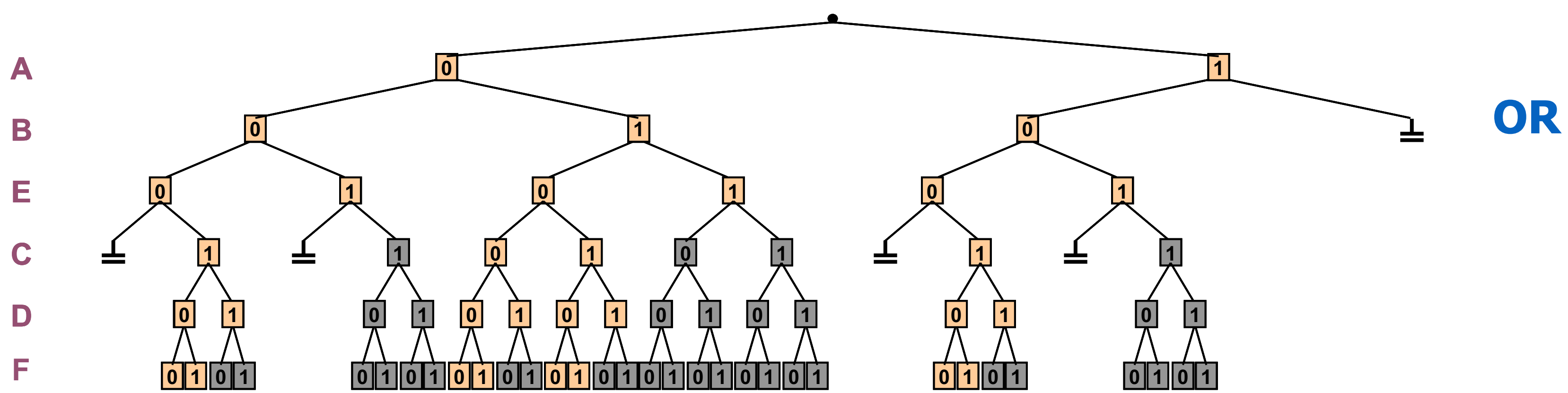
Random graphs with 20 nodes, 20 edges and 2 values per node.

(A=1,B=1)
(B=0,C=0)

AND/OR vs. OR

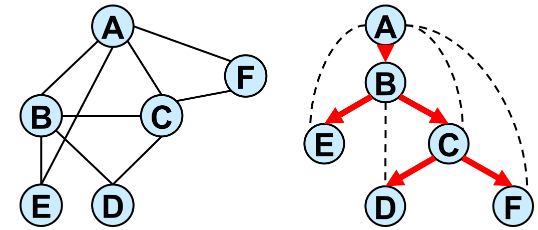


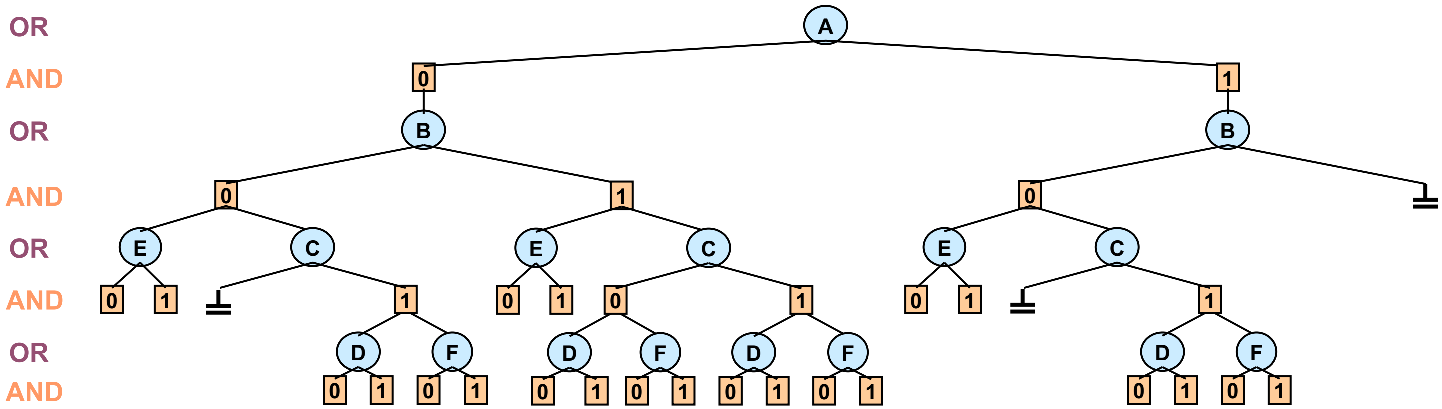




AND/OR vs. OR

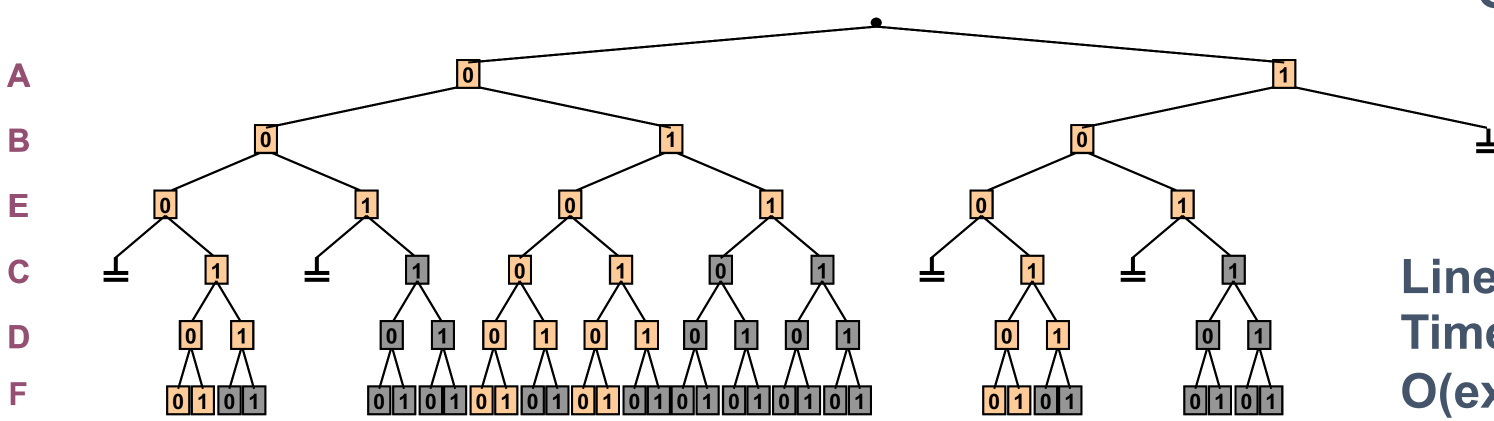
(A=1,B=1)
(B=0,C=0)





AND/OR

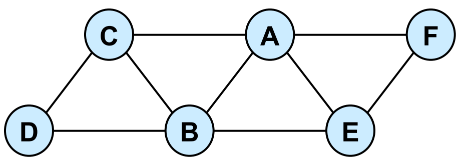
Space: linear
Time:
O(exp(m))
O(w* log n)



OR

Linear space,
Time:
O(exp(n))

#CSP – AND/OR search tree

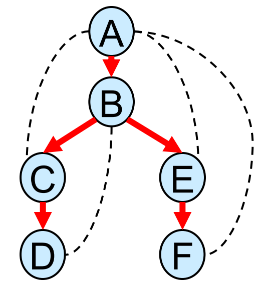


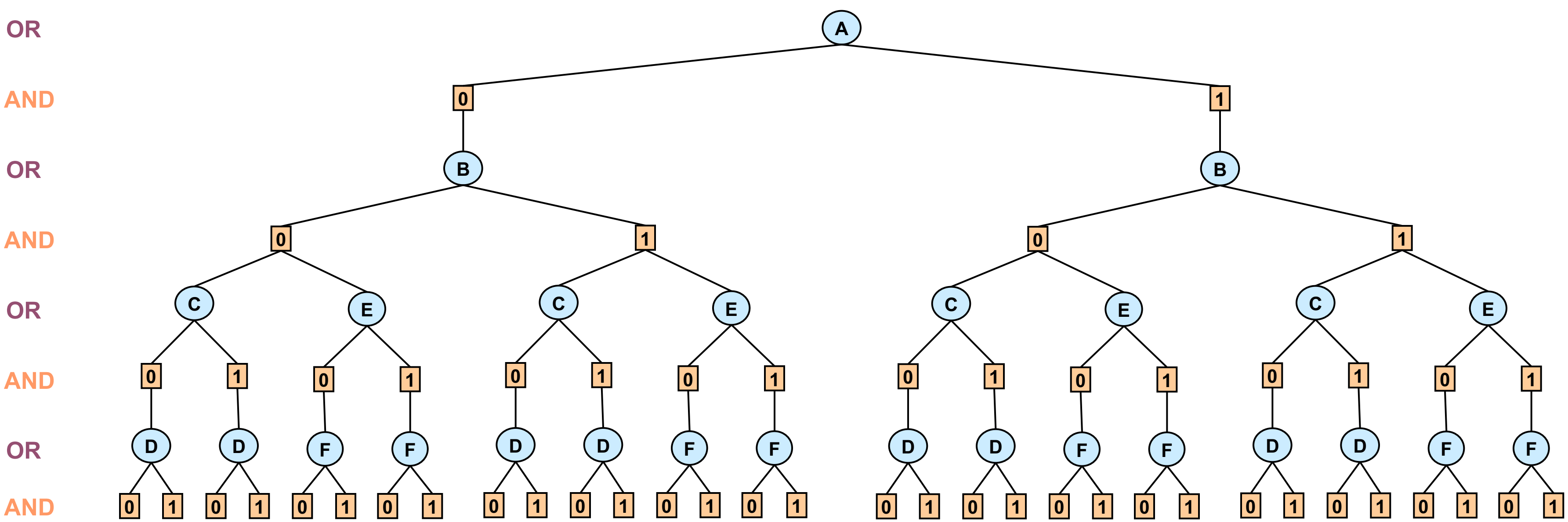
A	B	C	R_{ABC}
0	0	0	**1**
0	0	1	**1**
0	1	0	**0**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

B	C	D	R_{BCD}
0	0	0	**1**
0	0	1	**1**
0	1	0	**1**
0	1	1	**0**
1	0	0	**1**
1	0	1	**0**
1	1	0	**1**
1	1	1	**1**

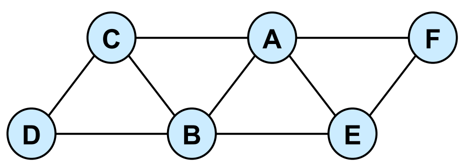
A	B	E	R_{ABE}
0	0	0	**1**
0	0	1	**0**
0	1	0	**1**
0	1	1	**1**
1	0	0	**0**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

A	E	F	R_{AEF}
0	0	0	**0**
0	0	1	**1**
0	1	0	**1**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**





#CSP – AND/OR search tree

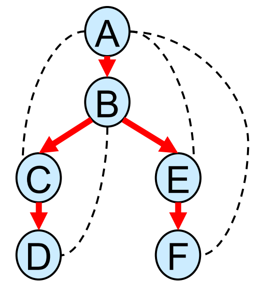


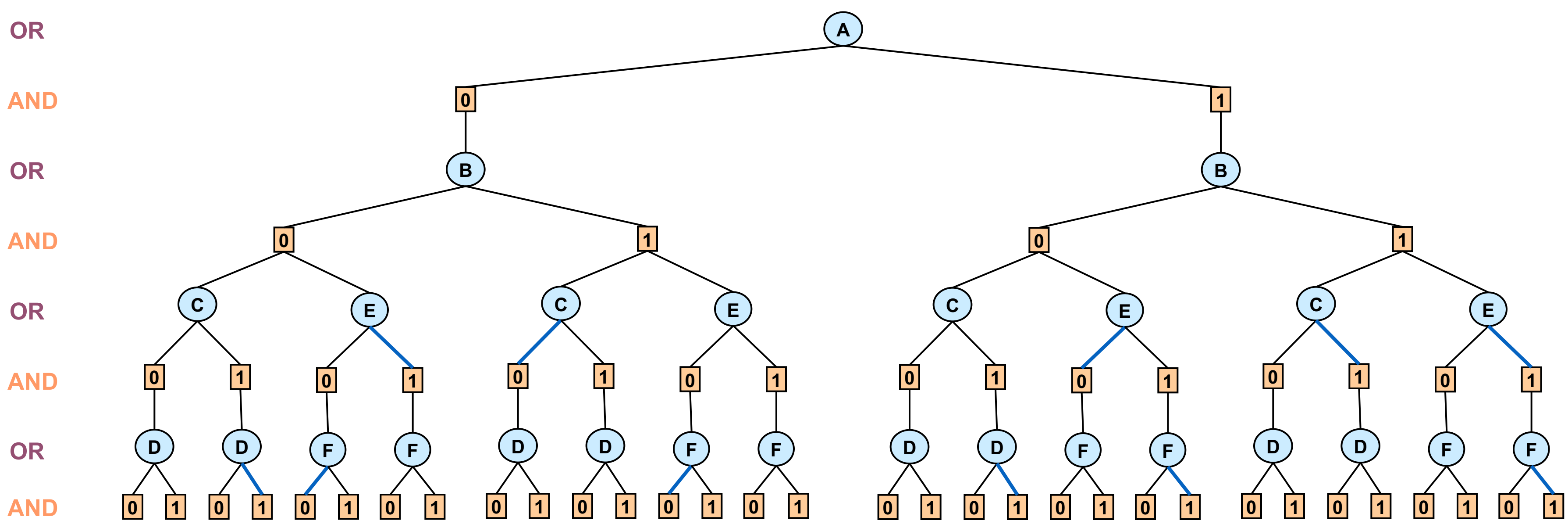
A	B	C	R_{ABC}
0	0	0	**1**
0	0	1	**1**
0	1	0	**0**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

B	C	D	R_{BCD}
0	0	0	**1**
0	0	1	**1**
0	1	0	**1**
0	1	1	**0**
1	0	0	**1**
1	0	1	**0**
1	1	0	**1**
1	1	1	**1**

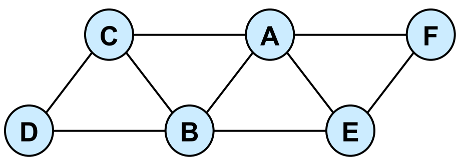
A	B	E	R_{ABE}
0	0	0	**1**
0	0	1	**0**
0	1	0	**1**
0	1	1	**1**
1	0	0	**0**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

A	E	F	R_{AEF}
0	0	0	**0**
0	0	1	**1**
0	1	0	**1**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**





#CSP – AND/OR Tree DFS

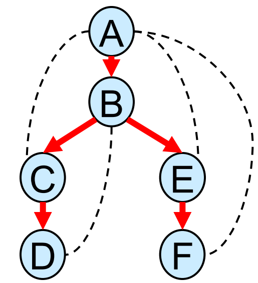


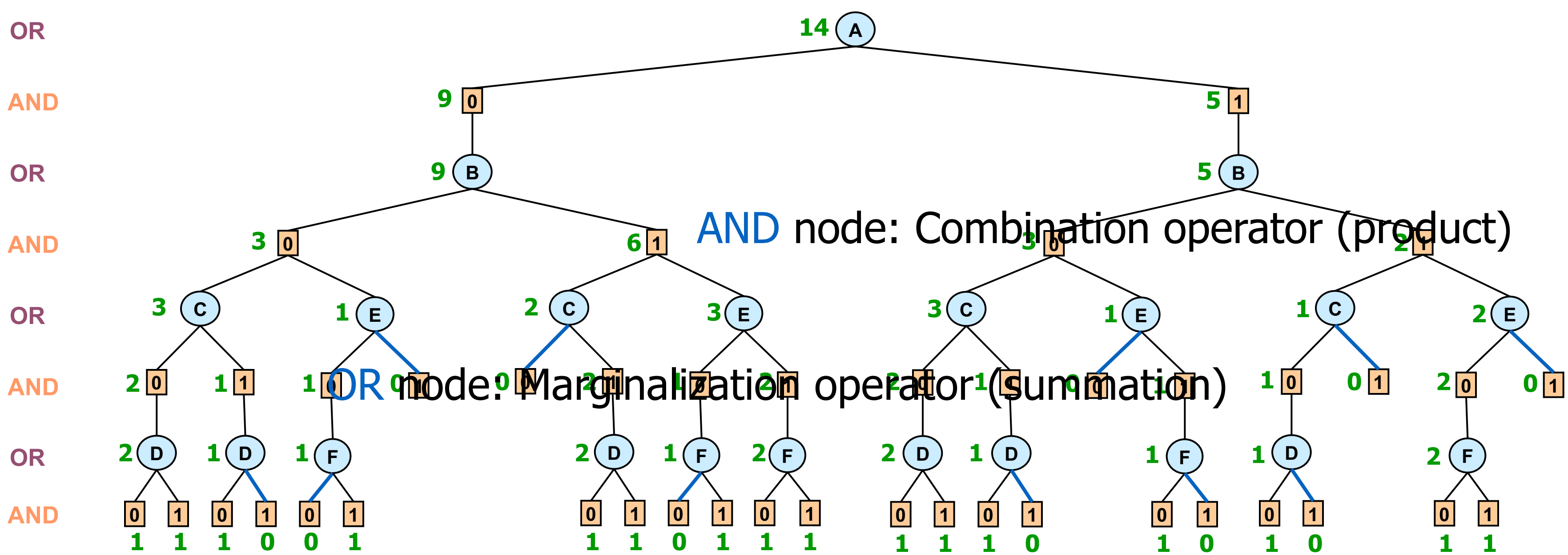
A	B	C	R_{ABC}
0	0	0	**1**
0	0	1	**1**
0	1	0	**0**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

B	C	D	R_{BCD}
0	0	0	**1**
0	0	1	**1**
0	1	0	**1**
0	1	1	**0**
1	0	0	**1**
1	0	1	**0**
1	1	0	**1**
1	1	1	**1**

A	B	E	R_{ABE}
0	0	0	**1**
0	0	1	**0**
0	1	0	**1**
0	1	1	**1**
1	0	0	**0**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

A	E	F	R_{AEF}
0	0	0	**0**
0	0	1	**1**
0	1	0	**1**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

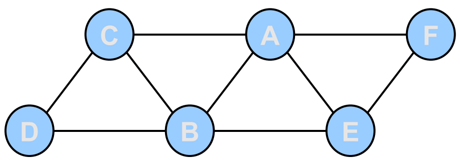


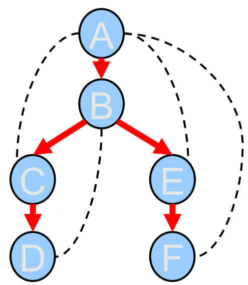


AND node: Combination operator (product)

OR node: Marginalization operator (summation)

AND/OR Tree search for COP





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

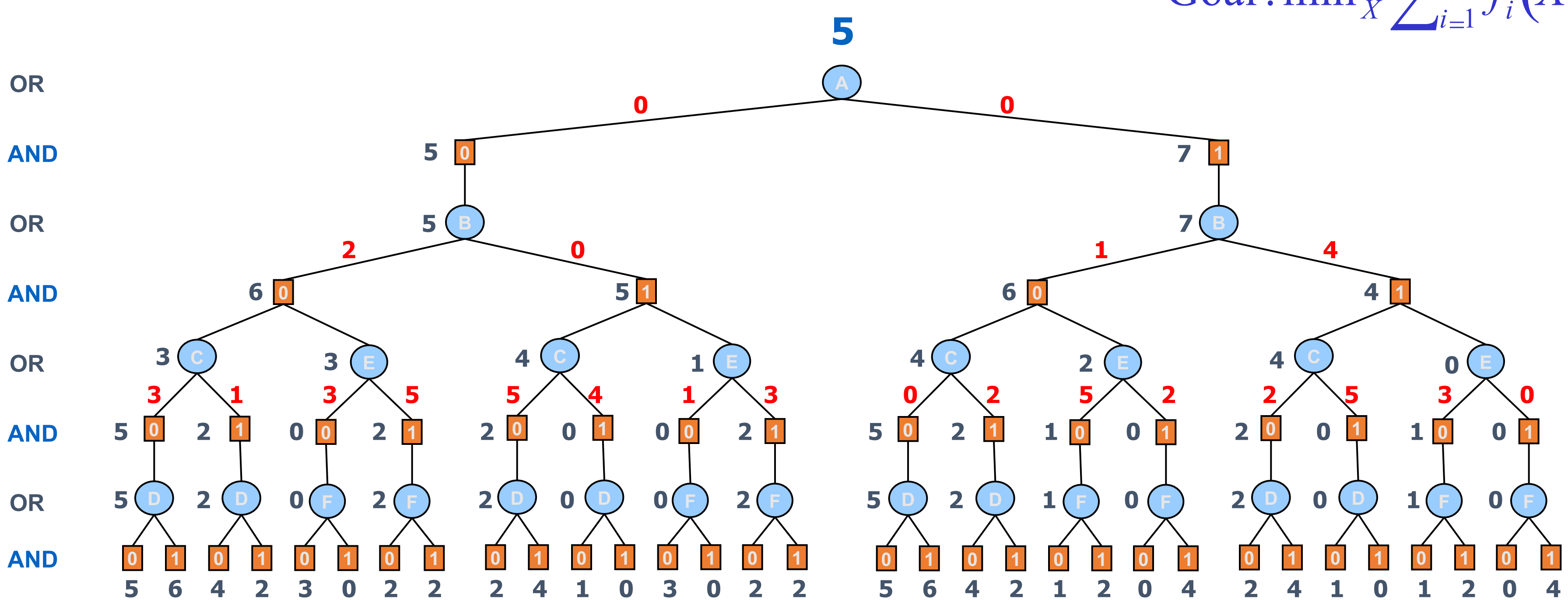
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

$$\text{Goal}: \min_X \sum_{i=1}^{9} f_i(X)$$



AND node = Combination operator (summation)

OR node = Marginalization operator (minimization)

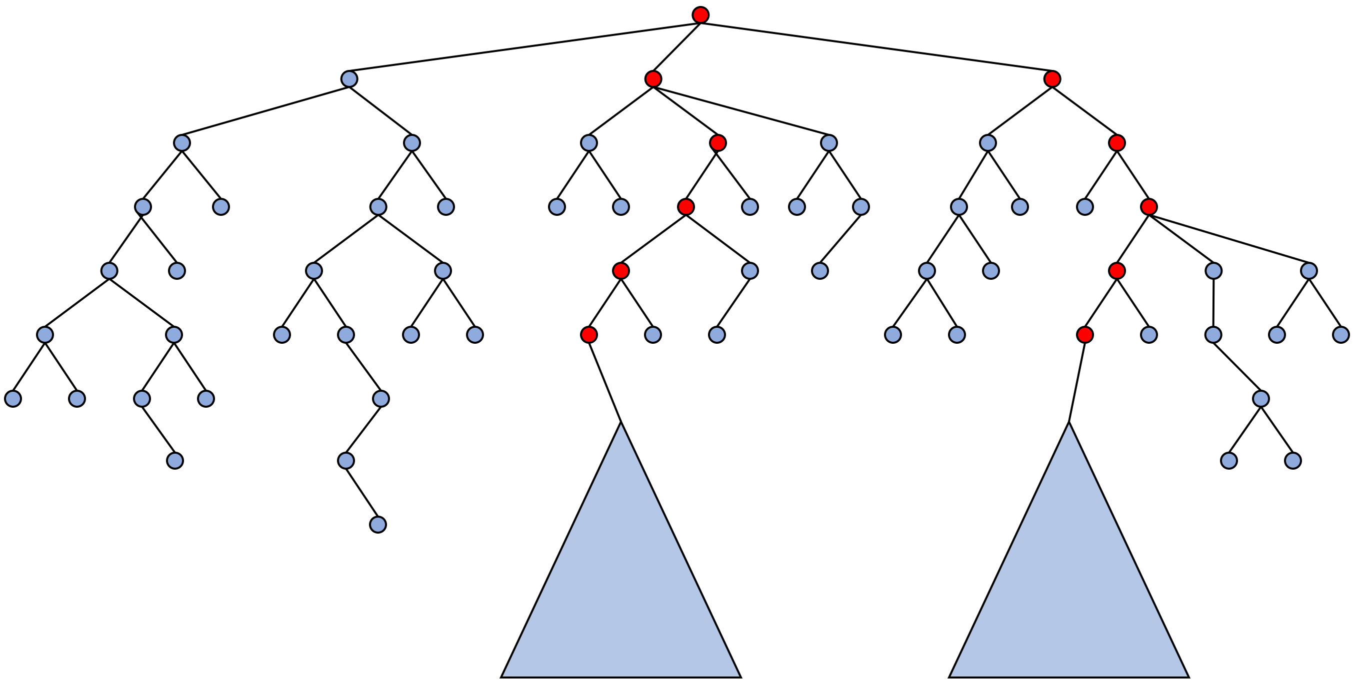
Summary of AND/OR search trees

- Based on a backbone pseudo-tree
- A solution is a subtree
- Each node has a **value** – cost of the optimal solution to the subproblem (computed recursively based on the values of the descendants)
- Solving a task = finding the value of the root node
- AND/OR search tree and algorithms are

  ([Freuder & Quinn85], [Collin, Dechter & Katz91], [Bayardo & Miranker95])
  - Space: O(n)
  - Time: O(exp(m)), where m is the depth of the pseudo-tree
  - Time: O(exp(w* log n))
  - BFS is time and space O(exp(w* log n)

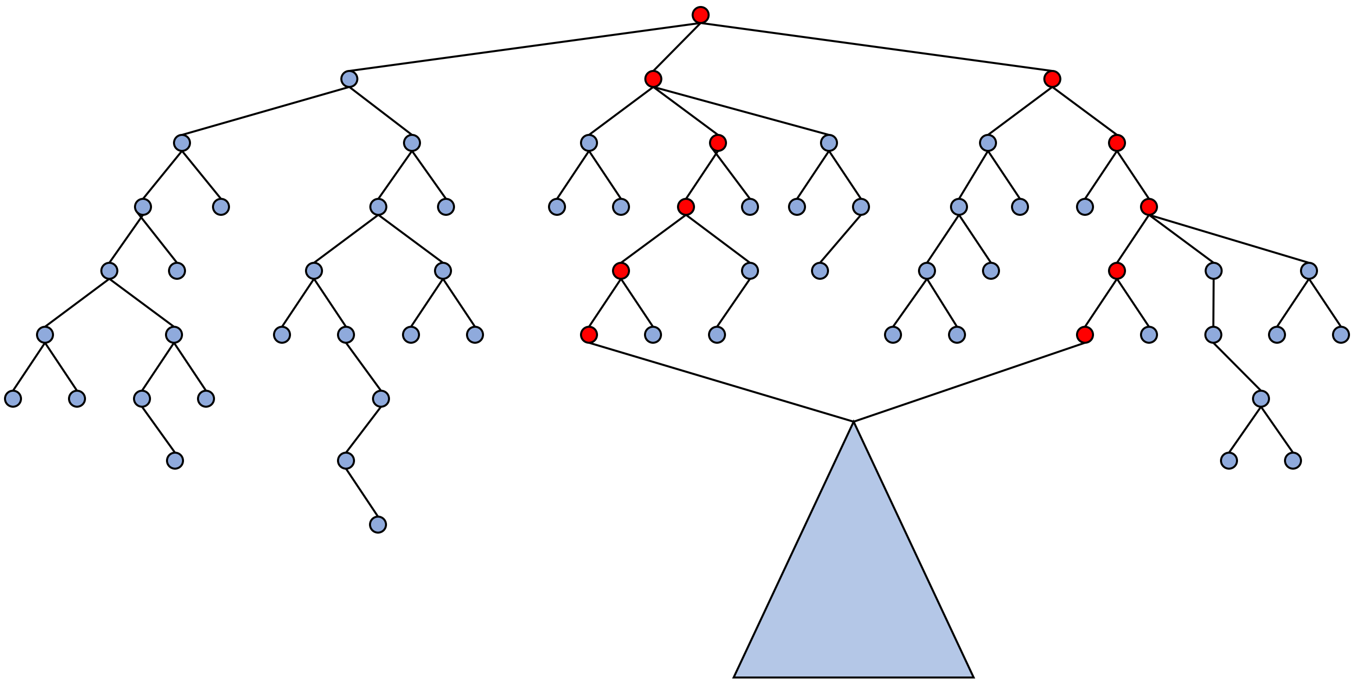
From search trees to search graphs

- Any two nodes that root identical subtrees or subgraphs can be merged

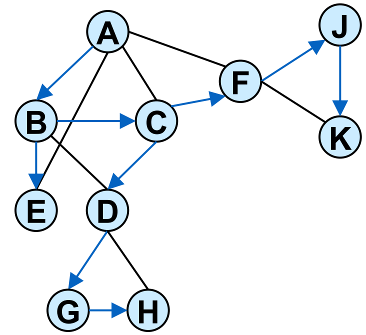


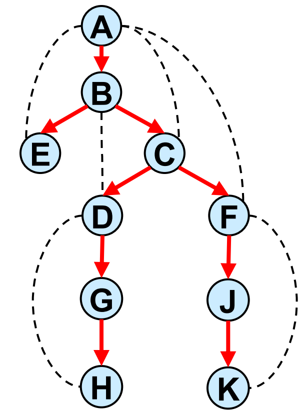
From search trees to search graphs

- Any two nodes that root identical subtrees or subgraphs can be merged



Caching



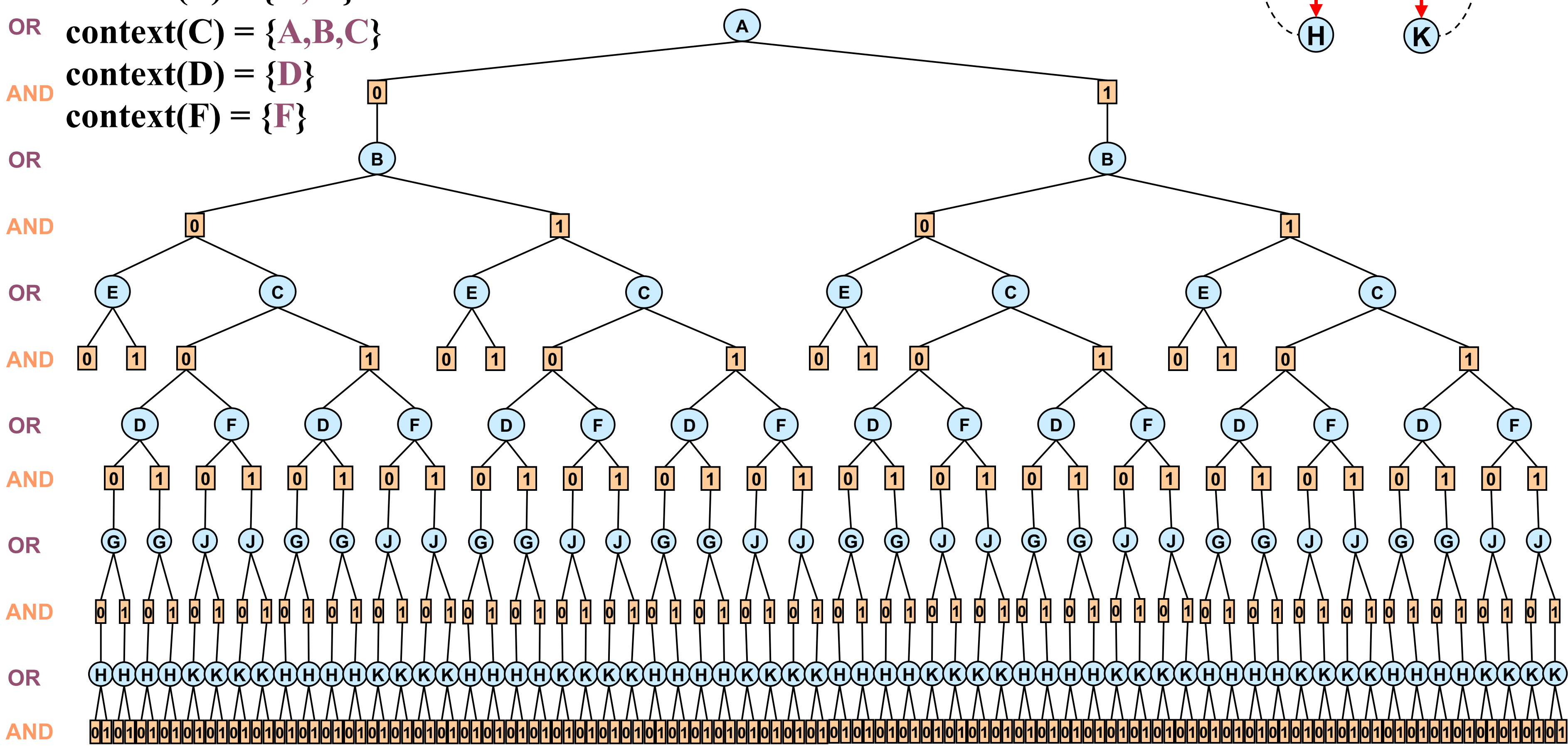


context(B) = {A, B}

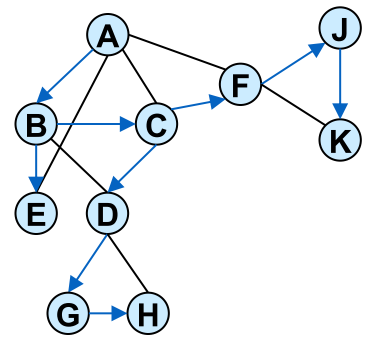
context(C) = {A,B,C}

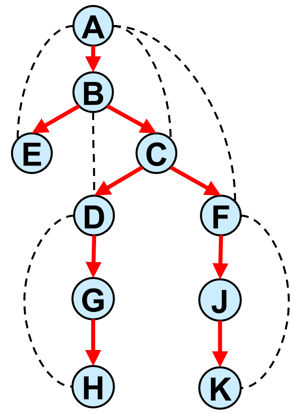
context(D) = {D}

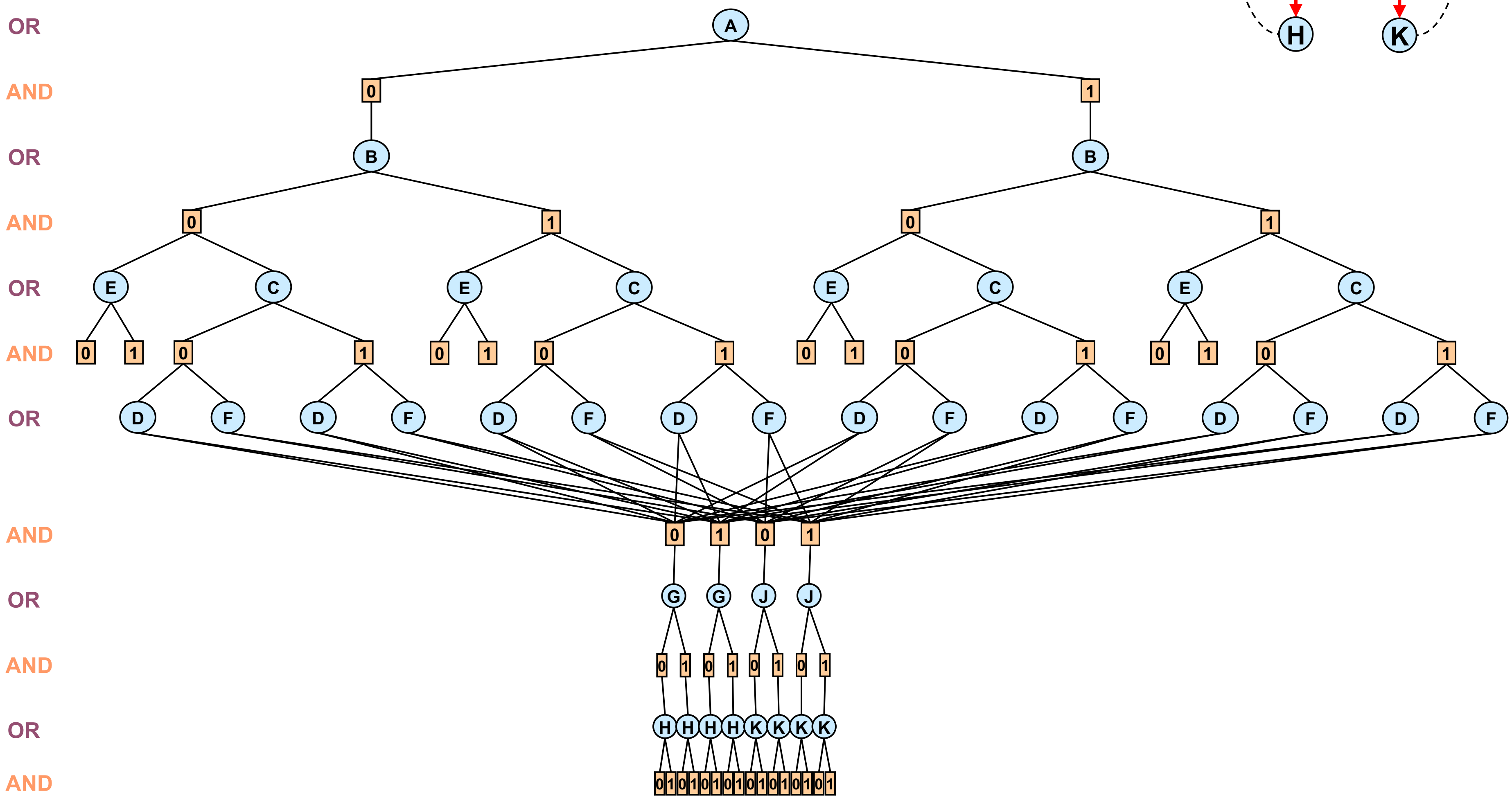
context(F) = {F}



An AND/OR graph: Caching goods



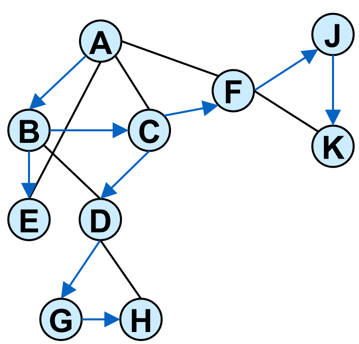


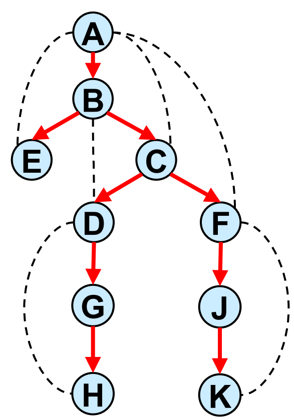


Context-based Caching

- Caching is possible when context is the same

- context = parent-separator set in induced pseudo-graph
  = current variable +
  parents connected to subtree below





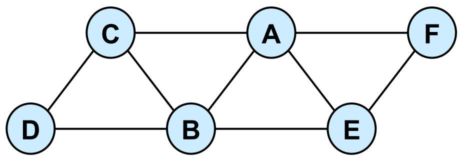
context(B) = {A, B}

context(c) = {A,B,C}

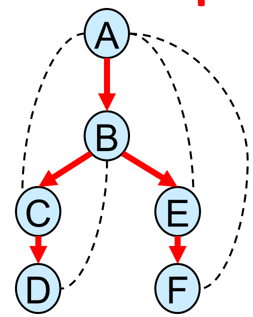
context(D) = {D}

context(F) = {F}

AND/OR Search Graph

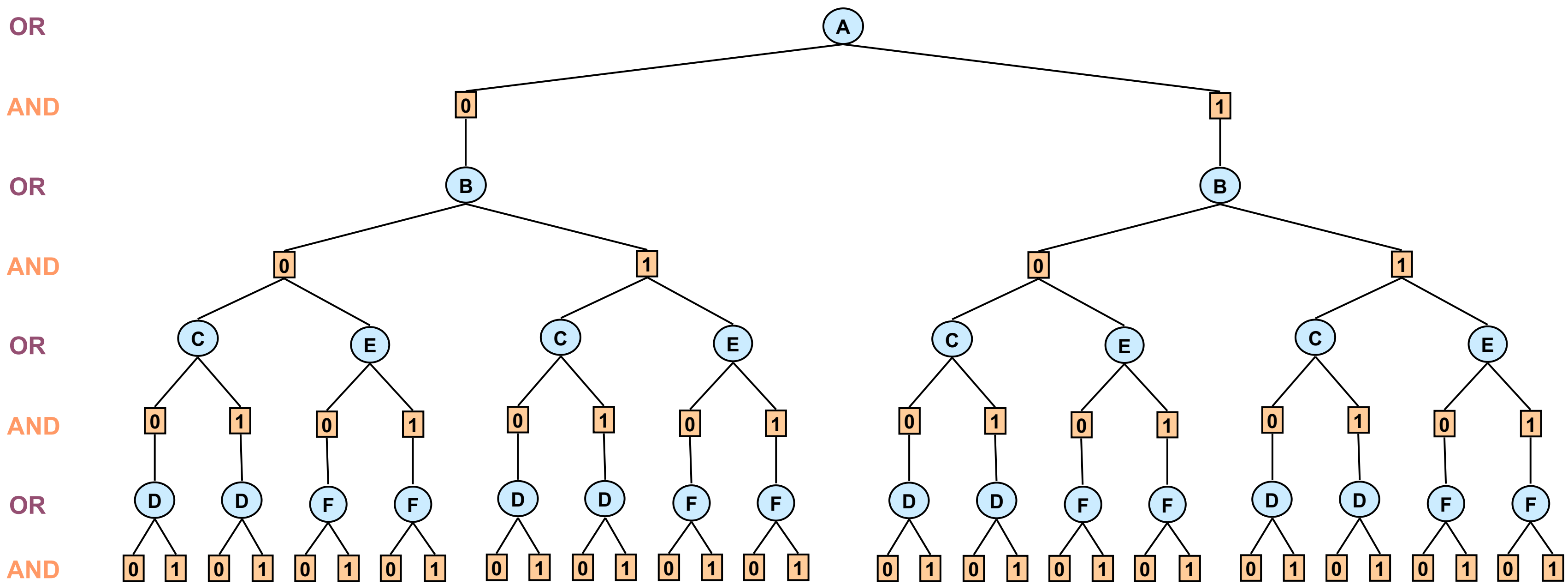


Primal graph

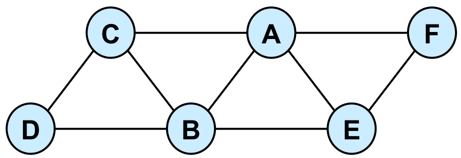


context(A) = {**A**}
context(B) = {**B,A**}
context(C) = {**C,B**}
context(D) = {**D**}
context(E) = {**E,A**}
context(F) = {**F**}

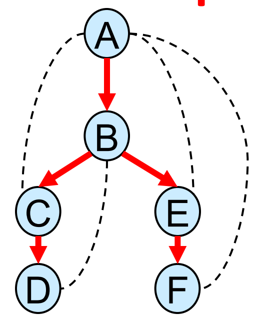
Pseudo-tree



AND/OR Search Graph

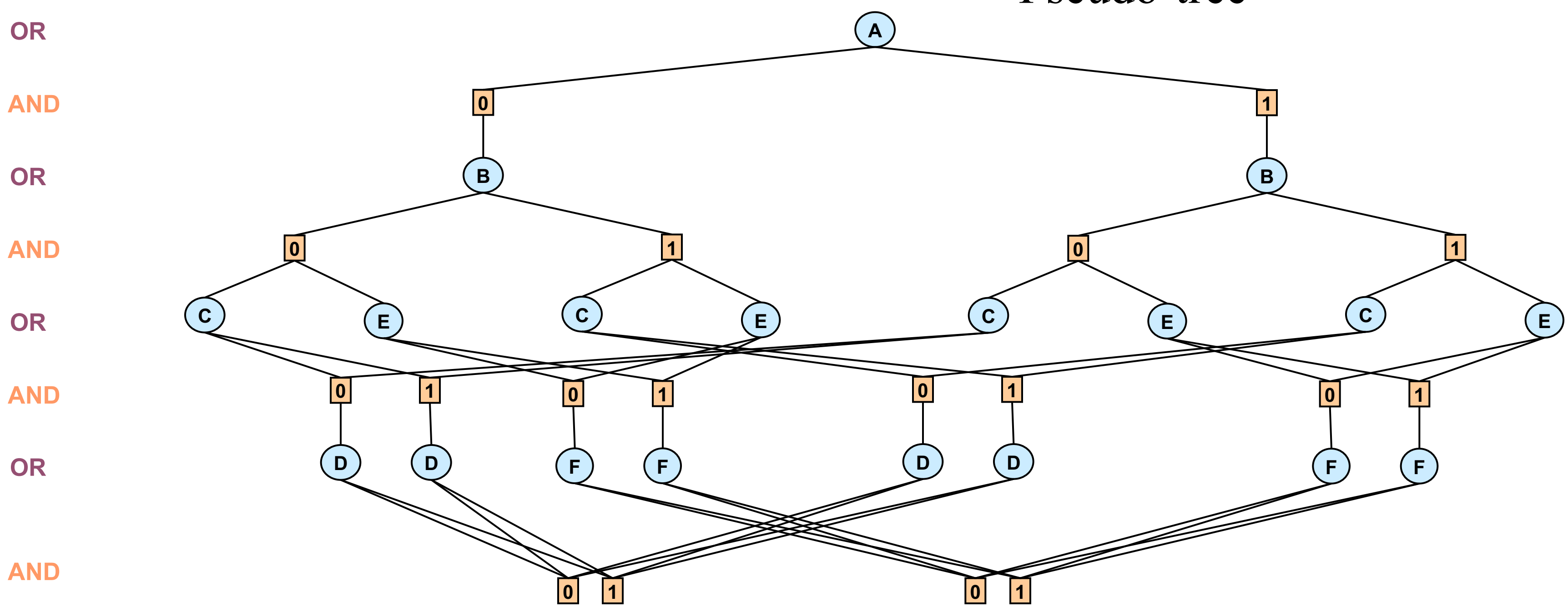


Primal graph

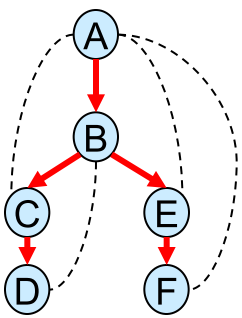


context(A) = {**A**}
context(B) = {**B,A**}
context(C) = {**C,B**}
context(D) = {**D**}
context(E) = {**E,A**}
context(F) = {**F**}

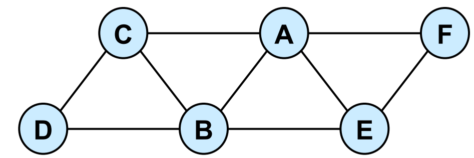
Pseudo-tree



Context-based caching



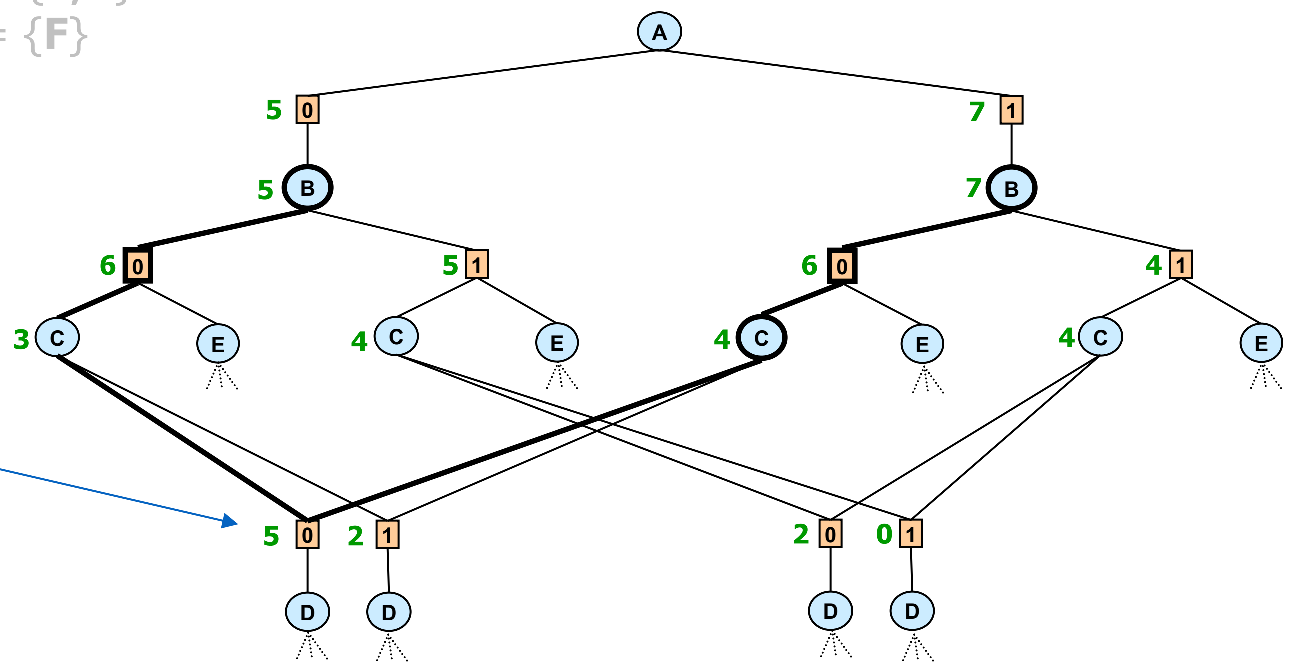
context(A) = {**A**}
context(B) = {**B,A**}
context(C) = {C,B}
context(D) = {**D**}
context(E) = {**E,A**}
context(F) = {**F**}



Primal graph

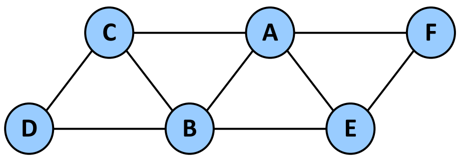
Cache Table (C)

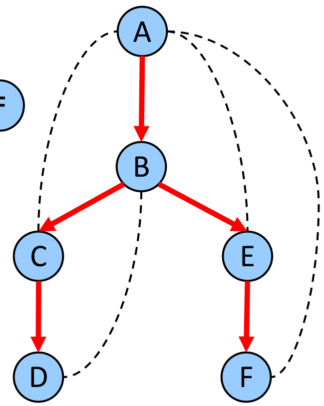
B	C	Value
0	0	5
0	1	2
1	0	2
1	1	0



Space: **O(exp(2))**

AND/OR Search Graph (Optimization)





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

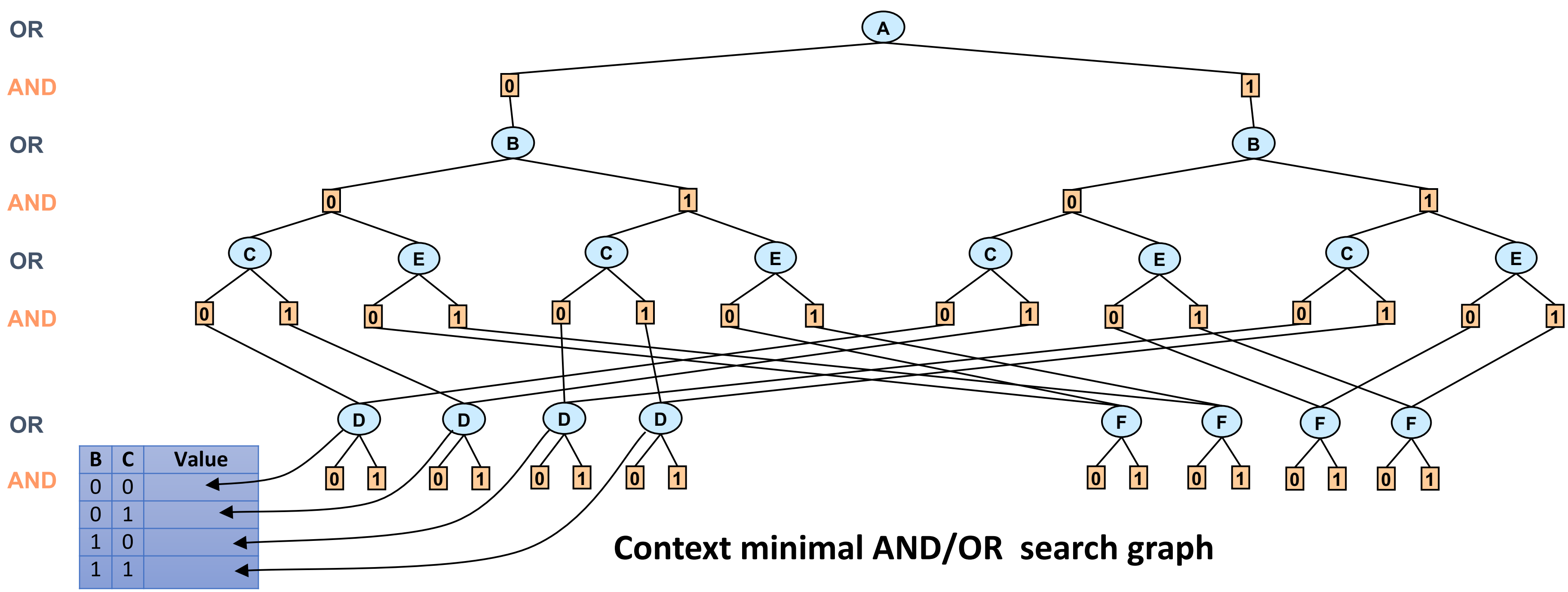
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

Objective function: $F^* = \min_x \sum_\alpha f_\alpha(x_\alpha)$



Context minimal AND/OR search graph

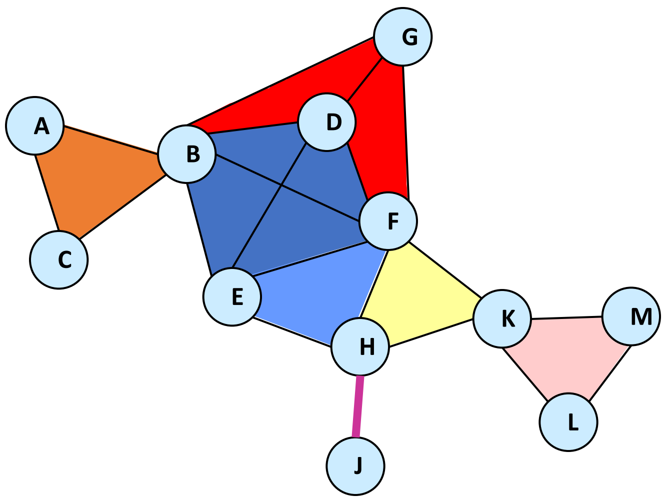
Cache table for **D**

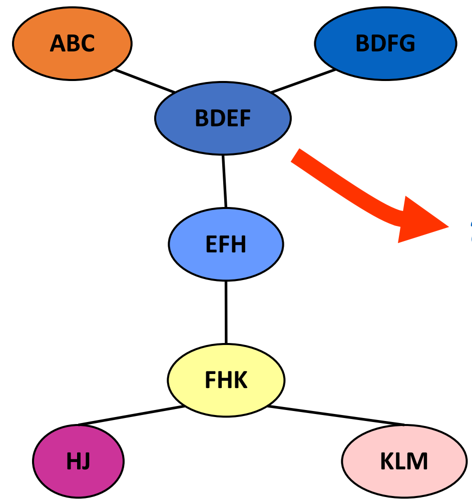
Complexity of AND/OR Graph

- **Theorem:** Traversing the AND/OR search graph is time and space exponential in the induced width/tree-width.

- If applied to the OR graph complexity is time and space exponential in the path-width.

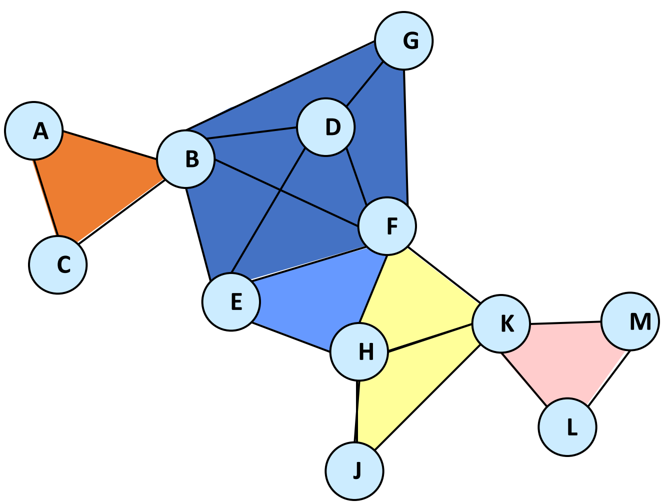
Treewidth vs. Pathwidth

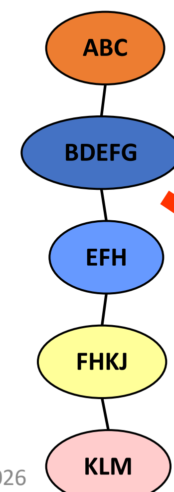




***treewidth* = 3**

= (max cluster size) - 1

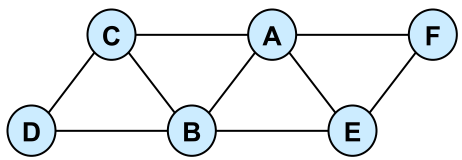




***pathwidth* = 4**

= (max cluster size) - 1

#CSP – AND/OR search tree

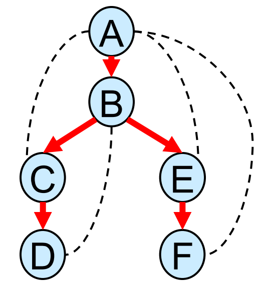


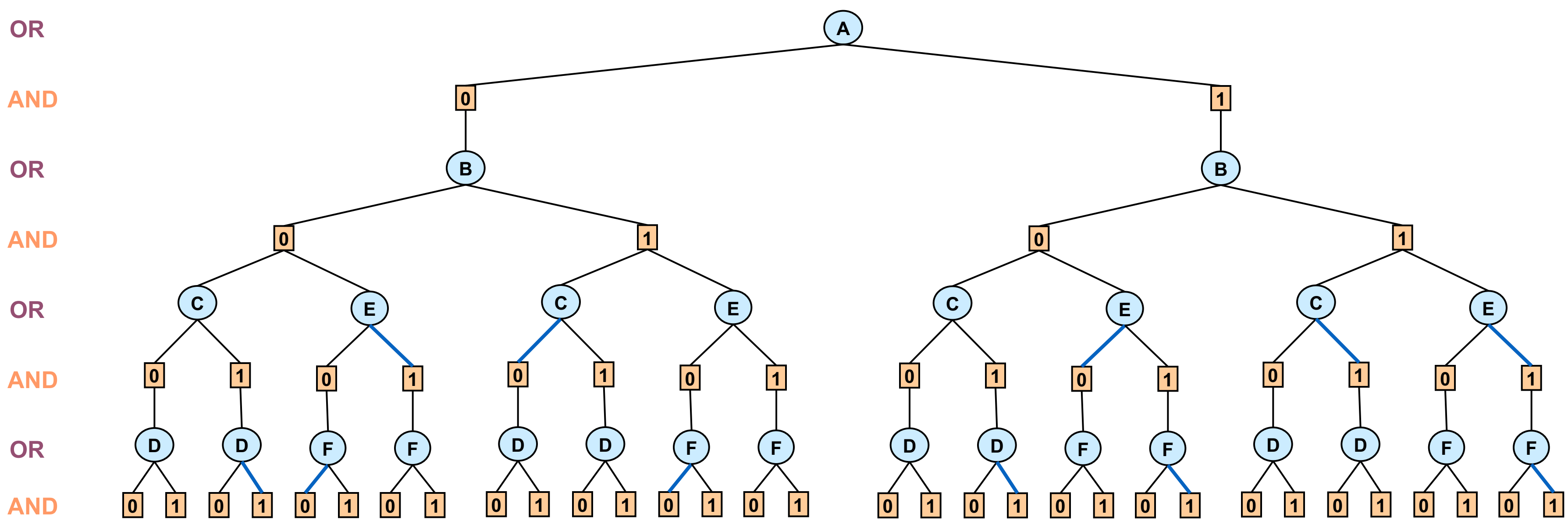
A	B	C	R_{ABC}
0	0	0	**1**
0	0	1	**1**
0	1	0	**0**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

B	C	D	R_{BCD}
0	0	0	**1**
0	0	1	**1**
0	1	0	**1**
0	1	1	**0**
1	0	0	**1**
1	0	1	**0**
1	1	0	**1**
1	1	1	**1**

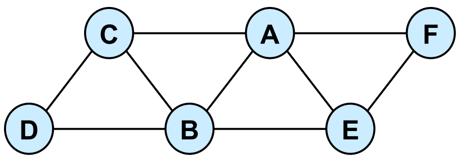
A	B	E	R_{ABE}
0	0	0	**1**
0	0	1	**0**
0	1	0	**1**
0	1	1	**1**
1	0	0	**0**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

A	E	F	R_{AEF}
0	0	0	**0**
0	0	1	**1**
0	1	0	**1**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**





#CSP – AND/OR tree dfs

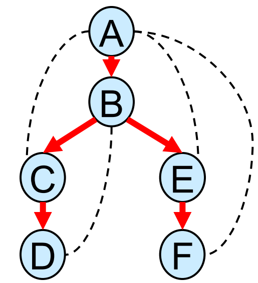


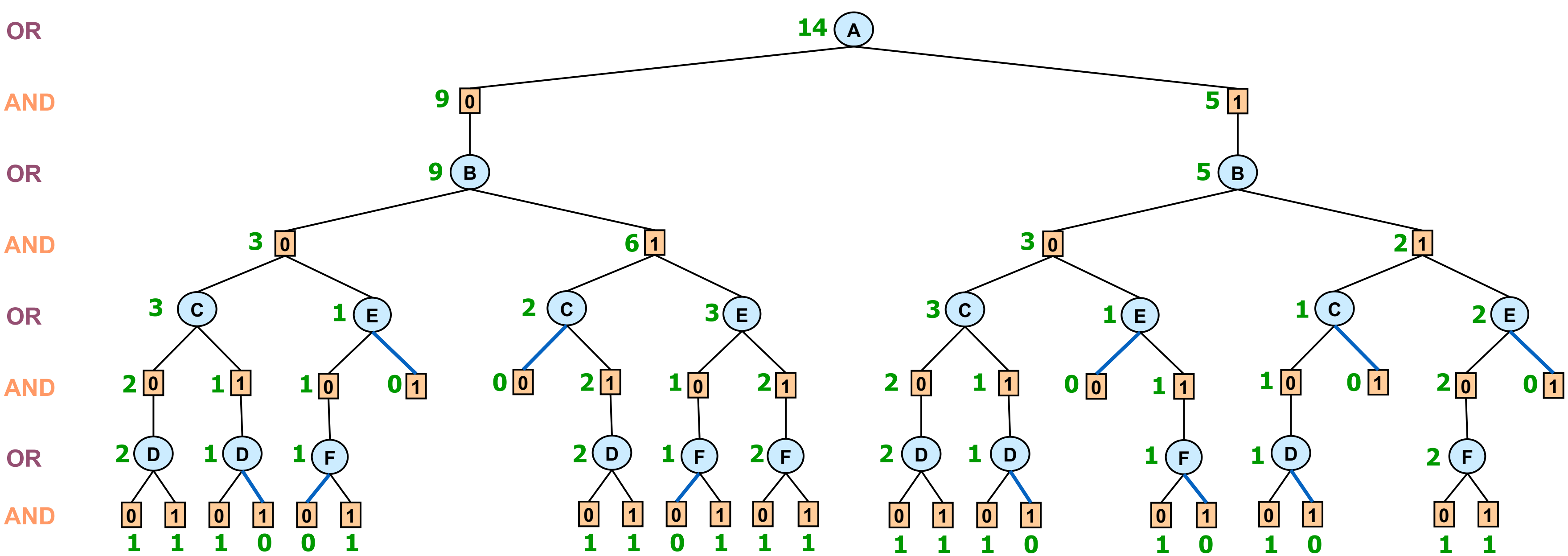
A	B	C	R_{ABC}
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

B	C	D	R_{BCD}
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

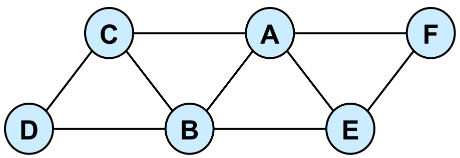
A	B	E	R_{ABE}
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

A	E	F	R_{AEF}
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0





#CSP – AND/OR search graph (Caching goods)

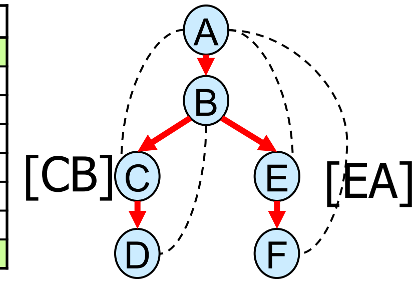


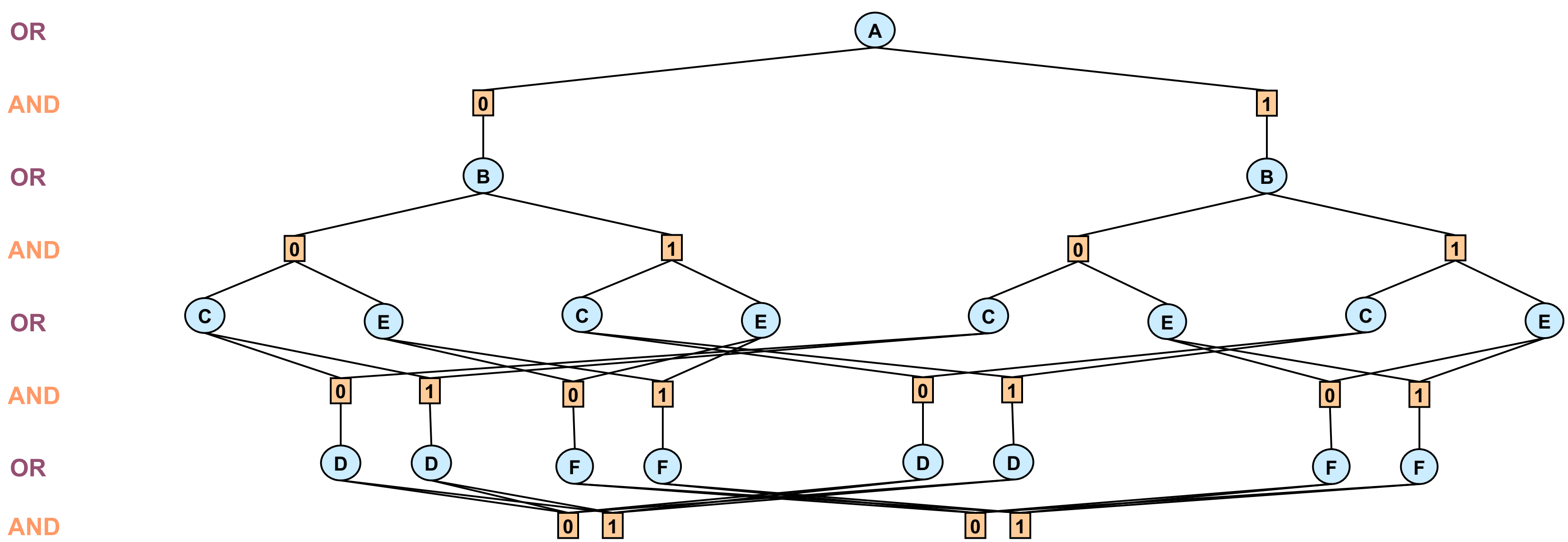
A	B	C	R_{ABC}
0	0	0	**1**
0	0	1	**1**
0	1	0	**0**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

B	C	D	R_{BCD}
0	0	0	**1**
0	0	1	**1**
0	1	0	**1**
0	1	1	**0**
1	0	0	**1**
1	0	1	**0**
1	1	0	**1**
1	1	1	**1**

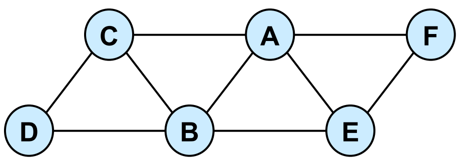
A	B	E	R_{ABE}
0	0	0	**1**
0	0	1	**0**
0	1	0	**1**
0	1	1	**1**
1	0	0	**0**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

A	E	F	R_{AEF}
0	0	0	**0**
0	0	1	**1**
0	1	0	**1**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**





#CSP – AND/OR search graph (Caching goods)

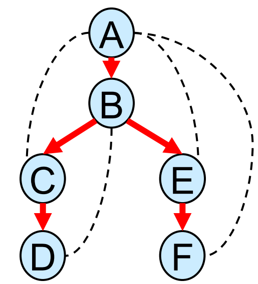


A	B	C	R_{ABC}
0	0	0	**1**
0	0	1	**1**
0	1	0	**0**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

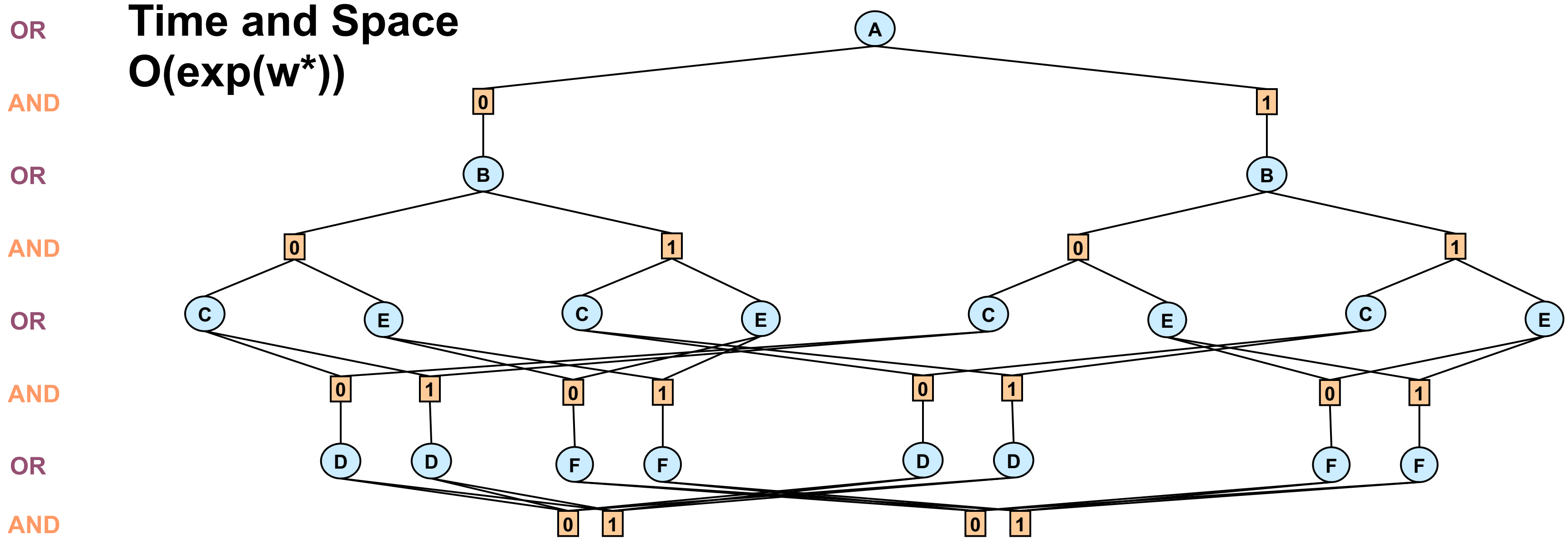
B	C	D	R_{BCD}
0	0	0	**1**
0	0	1	**1**
0	1	0	**1**
0	1	1	**0**
1	0	0	**1**
1	0	1	**0**
1	1	0	**1**
1	1	1	**1**

A	B	E	R_{ABE}
0	0	0	**1**
0	0	1	**0**
0	1	0	**1**
0	1	1	**1**
1	0	0	**0**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

A	E	F	R_{AEF}
0	0	0	**0**
0	0	1	**1**
0	1	0	**1**
0	1	1	**1**
1	0	0	**1**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

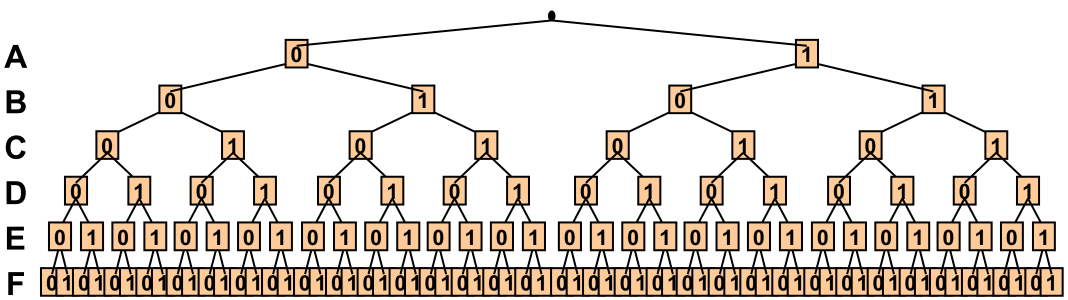


Time and Space O(exp(w*))



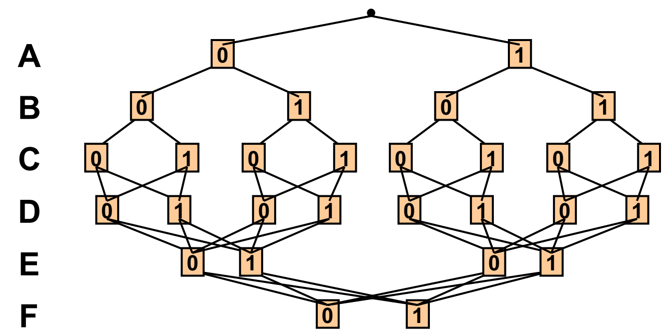
Space O(exp(sep-w*))

All Four Search Spaces



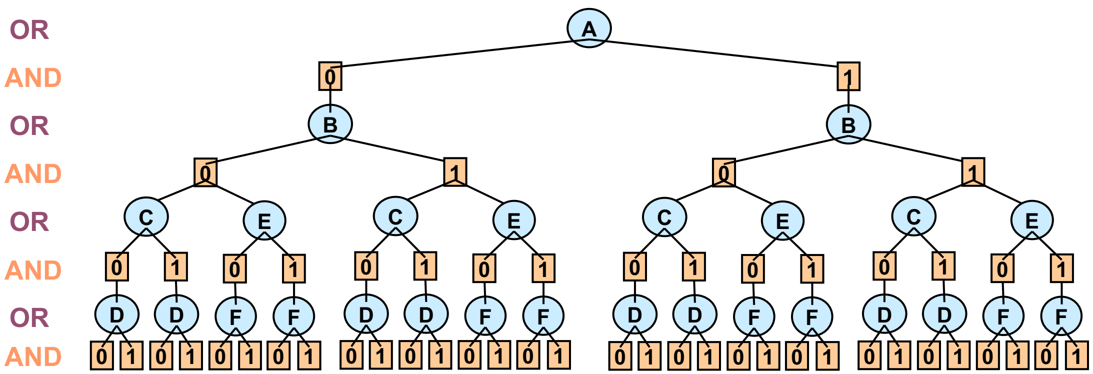
Full OR search tree

126 nodes



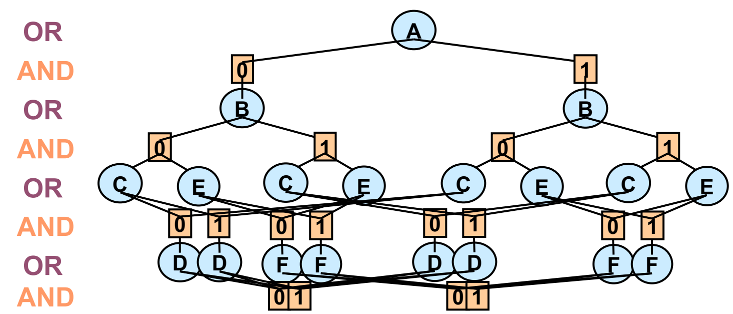
Context minimal OR search graph

28 nodes



Full AND/OR search tree

54 AND nodes



Context minimal AND/OR search graph

18 AND nodes

AND/OR vs. OR dfs algorithms

k = domain size
m = pseudo-tree depth
n = number of variables
w^*= induced width
pw^*= path width

- AND/OR tree
  - Space: $O(n)$
  - Time: $O(n\ k^m)$
    $O(n\ k^{w^* \log n})$

  (Freuder85; Bayardo95; Darwiche01)

- OR tree
  - Space: $O(n)$
  - Time: $O(k^n)$

- AND/OR graph
  - Space: $O(n\ k^{w^*})$
  - Time: $O(n\ k^{w^*})$

- OR graph
  - Space: $O(n\ k^{pw^*})$
  - Time: $O(n\ k^{pw^*})$

Searching AND/OR graphs

- AO(i): searches depth-first, cache i-context
  - i = the max size of a cache table (i.e. number of variables in a context)

i=0 → i=w*

Space: O(n)

Time: O(exp(w* log n))

Space: O(exp w*)

Time: O(exp w*)

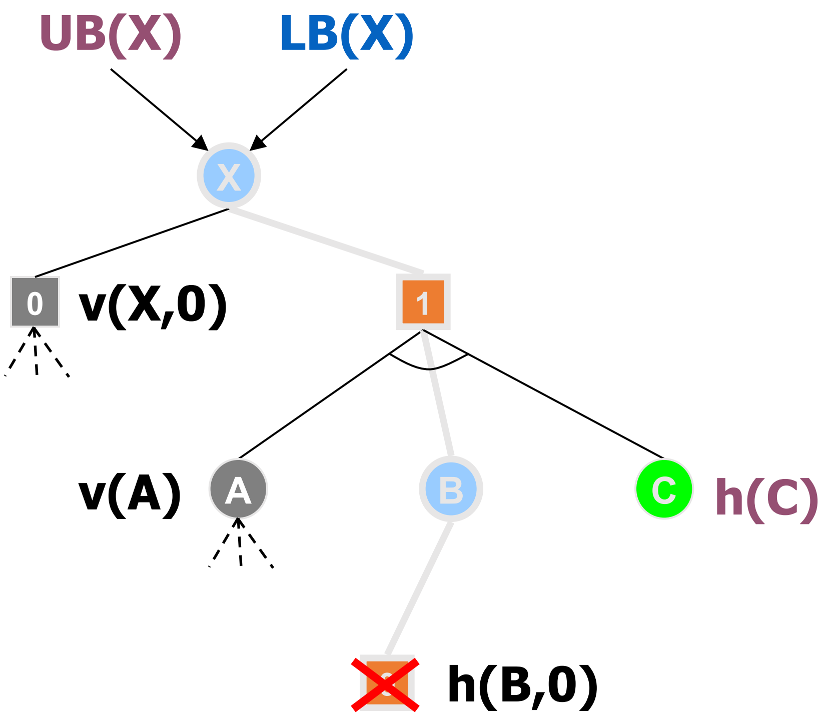
AO(i) time complexity?

AND/OR branch-and-bound (AOBB)

- Associate each node **n** with a static heuristic estimate h(n) of v(n)
  - h(n) is a lower bound on the value v(n)

- For every node **n** in the search tree:
  - ub(n) – current best solution cost rooted at n
  - lb(n) – lower bound on the minimal cost at n

Lower/Upper bounds



UB(X) = best cost below X (i.e. v(X,0))

LB(X) = LB(X,1)

LB(X,1) = l(X,1) + v(A) + h(C) + LB(B)

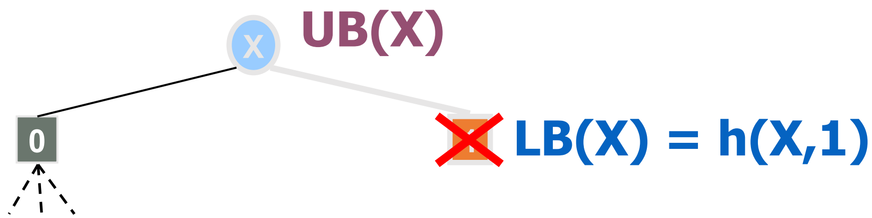
LB(B) = LB(B,0)

LB(B,0) = h(B,0)

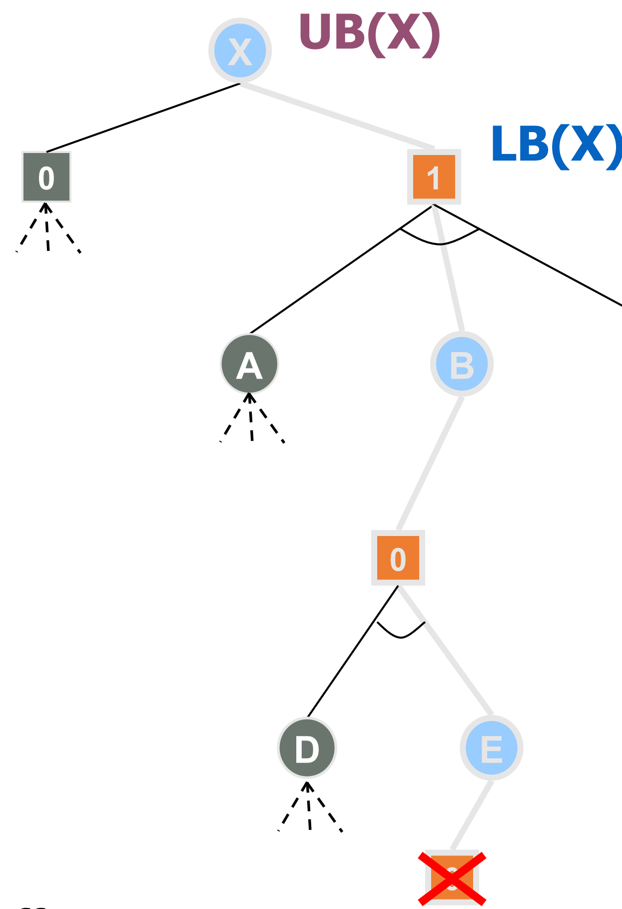
Prune below AND node (B,0) if LB(X) ≥ UB(X)

Shallow/deep cutoffs

Prune if LB(X) ≥ UB(X)



Shallow cutoff



Deep cutoff

Reminiscent of **Minimax** shallow/deep cutoffs

Summary of AOBB

- Traverses the AND/OR search tree in a depth-first manner
- Lower bounds computed based on heuristic estimates of nodes at the frontier of search, as well as the values of nodes already explored
- Prunes the search space as soon as an upper-lower bound violation occurs

Heuristics for AND/OR

- In the AND/OR search space **h(n)** can be computed using any heuristic. We used:
  - Static Mini-Bucket heuristics
  - Dynamic Mini-Bucket heuristics
  - Maintaining FDAC [Larrosa & Schiex03]
    (full directional soft arc-consistency)

Empirical evaluation

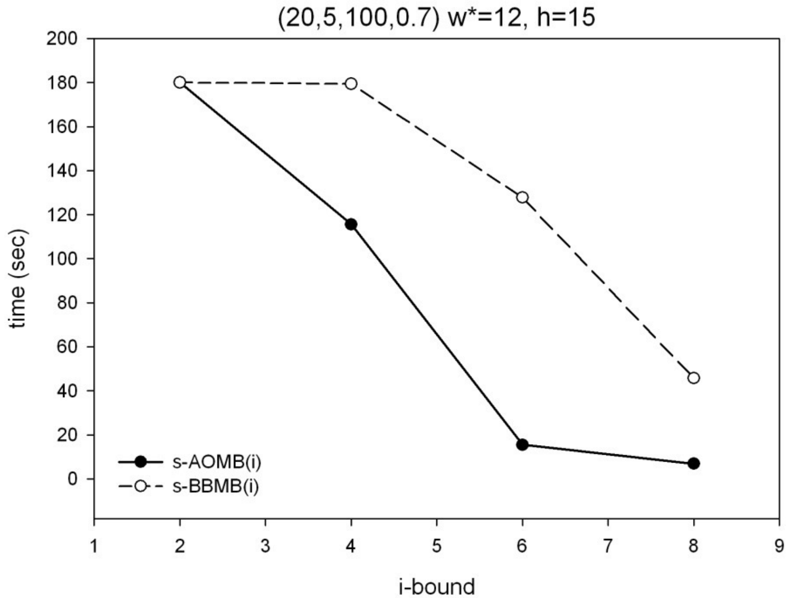
- Tasks
  - Solving WCSPs
  - Finding the MPE in belief networks

- Benchmarks (WCSP)
  - Random binary WCSPs
  - RLFAP networks (CELAR6)
  - Bayesian Networks Repository

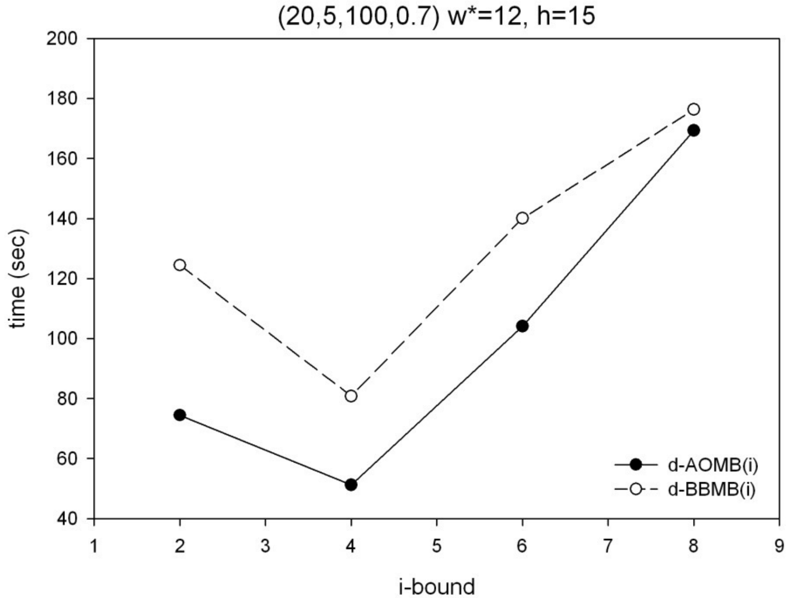
- Algorithms
  - s-AOMB(i), d-AOMB(i), AOMFDAC
  - s-BBMB(i), d-BBMB(i), BBMFDAC
  - Static variable ordering (dfs traversal of the pseudo-tree)

Random binary wcsps
(Marinescu and Dechter, 2005)

S-AOMB vs S-BBMB



D-AOMB vs D-BBMB

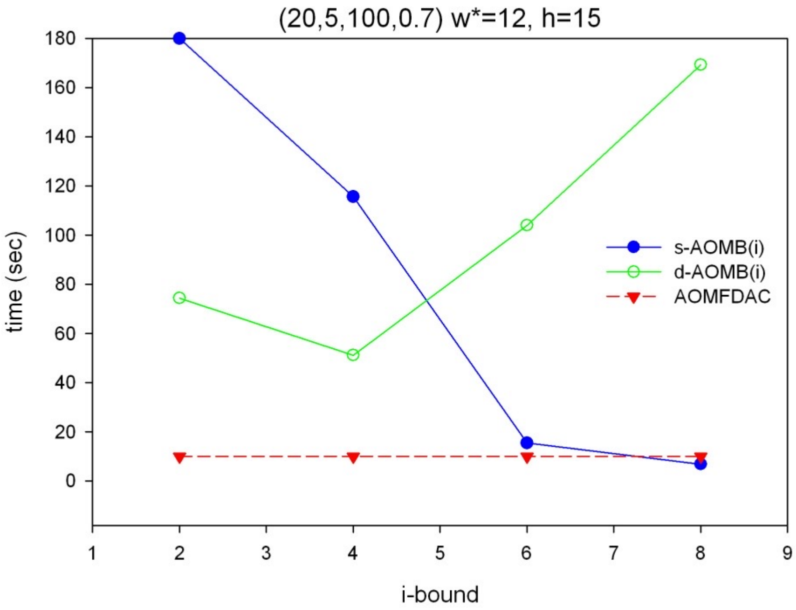


Random networks with n=20 (number of variables), d=5 (domain size), c=100 (number of constraints), t=70% (tightness). Time limit 180 seconds.

AO search is superior to OR search

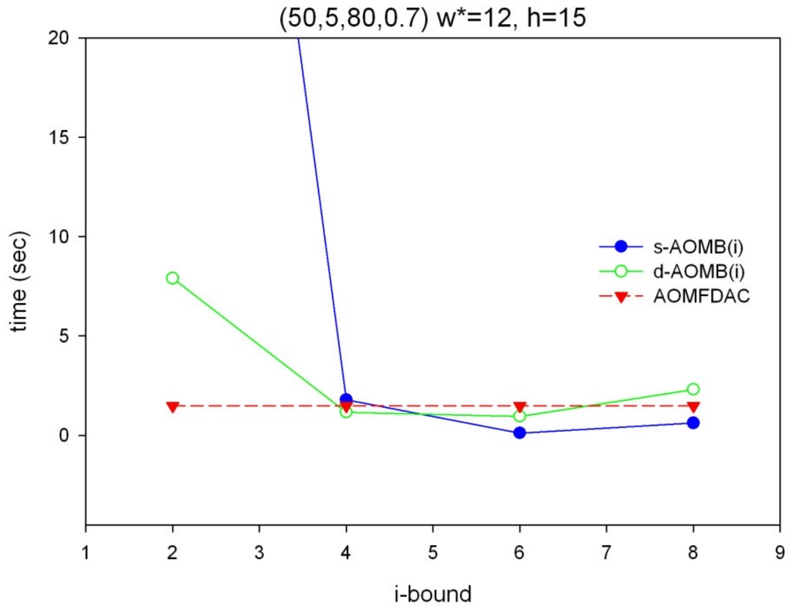
Random binary wcsps (contd.)

dense



n=20 (variables), d=5 (domain size), c=100 (constraints), t=70% (tightness)

sparse



n=50 (variables), d=5 (domain size), c=80 (constraints), t=70% (tightness)

AOMB for large i is competitive with AOMFDAC

Resource allocation

Radio Link Frequency Assignment Problem (RLFAP)

Instance	BBMFDAC		AOMFDAC	
	time (sec)	nodes	time (sec)	nodes
CELAR6-SUB0	2.78	1,871	**1.98**	435
CELAR6-SUB1	2,420.93	364,986	**981.98**	180,784
CELAR6-SUB2	8,801.12	19,544,182	**1,138.87**	175,377
CELAR6-SUB3	38,889.20	91,168,896	**4,028.59**	846,986
CELAR6-SUB4	84,478.40	6,955,039	**47,115.40**	4,643,229

CELAR6 sub-instances

AOMFDAC is superior to **OR**MFDAC

Bayesian networks repository

Network (n,d,w*,h)	Algorithm	i=2 time	i=2 nodes	i=3 time	i=3 nodes	i=4 time	i=4 nodes	i=5 time	i=5 nodes
Barley (48,67,7,17)	s-AOMB(i)	-	8.5M	-	7.6M	46.22	807K	**0.563**	9.6K
	s-BBMB(i)	-	16M	-	18M	-	17M	-	14M
	d-AOMB(i)	-	79K	136.0	23K	12.55	667	45.95	567
	d-BBMB(i)	-	2.2M	-	1M	346.1	76K	-	86K
Munin1 (189,21,11,24)	s-AOMB(i)	57.36	1.2M	12.08	260K	7.203	172K	**1.657**	43K
	s-BBMB(i)	-	8.5M	-	9M	-	10M	-	8M
	d-AOMB(i)	66.56	185K	12.47	8.1K	10.30	1.6K	11.99	523
	d-BBMB(i)	-	405K	-	430K	-	235K	14.63	917
Munin3 (1044,21,7,25)	s-AOMB(i)	-	5.9M	-	4.9M	1.313	17K	**0.453**	6K
	s-BBMB(i)	-	1.4M	-	1.2M	-	316K	-	1.5M
	d-AOMB(i)	-	2.3M	68.64	58K	3.594	5.9K	2.844	3.8K
	d-BBMB(i)	-	33K	-	125K	-	52K	-	31K

Time limit 600 seconds

available at http://www.cs.huji.ac.il/labs/compbio/Repository

Static AO is better with accurate heuristic (large i)

Outline

- **Introduction**
  - Optimization tasks for graphical models
  - Solving optimization problems with inference and search
- **Inference**
  - Bucket elimination, dynamic programming
  - Mini-bucket elimination
- **Search**
  - Branch and bound and best-first
  - Lower-bounding heuristics
  - AND/OR search spaces